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Explicit expressions of $\boldsymbol{S}(v)$, $\boldsymbol{H}(v)$ and $\boldsymbol{L}(v)$ for anisotropic elastic materials

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Abstract

The three matrices $\boldsymbol{L}(v)$, $\boldsymbol{S}(v)$ and $\boldsymbol{H}(v)$, appearing frequently in the investigations of the two-dimensional steady state motions of elastic solids, are expressed explicitly in terms of the elastic stiffness for general anisotropic materials. The special cases of monoclinic materials with a plane of symmetry at $x_3 = 0$, $x_1 = 0$, and $x_2 = 0$ are all deduced. Results for orthotropic materials appearing in the literature may be recovered from the present explicit expressions.

Keywords: Stroh formalism; Steady state motion; Anisotropic elasticity; Monoclinic materials; Orthotropic materials

1. Introduction

It is known that Stroh formalism is mathematically elegant and technically powerful in determining the two-dimensional deformations not only in *anisotropic elastostatics* (Stroh, 1958; Ting, 1996) but also in *anisotropic elastodynamics* (Ting, 1996). In elastostatics, the three real matrices $\boldsymbol{L}$, $\boldsymbol{S}$ and $\boldsymbol{H}$, called *Barnett–Lothe tensors* appear often in the solutions of many anisotropic boundary value problems. Due to their importance, the explicit expressions of Barnett–Lothe tensors have been investigated by many researchers. The most general anisotropic materials without any symmetry plane assumed were considered by Wei and Ting (1994) and Ting (1997). Other related works for anisotropic elastic materials with special symmetry plane assumed are Dongye and Ting (1989) for orthotropic materials, Ting (1992) and Suo (1990) for monoclinic materials with the symmetry plane at $x_3 = 0$, Tanuma (1996), and Nakamura and Tanuma (1996) for transversely isotropic materials.

In elastodynamic problems, if the elastic body is in a steady state motion in a certain direction with a constant speed $v > 0$, then the solutions of these problems are often related to three matrices $\boldsymbol{L}(v)$, $\boldsymbol{S}(v)$ and $\boldsymbol{H}(v)$.

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Explicit expressions of these three matrices are also useful in the derivation of the secular equation when surface wave speed in an anisotropic elastic half-space is concerned (Ting, 2002). These matrices reduce, respectively, to *Barnett–Lothe tensors* $\boldsymbol{L}$, $\boldsymbol{S}$ and $\boldsymbol{H}$ when $v = 0$. For orthotropic materials, Dongye and Ting (1989) presented the explicit expressions of $\boldsymbol{L}(v)$, $\boldsymbol{S}(v)$ and $\boldsymbol{H}(v)$. Chadwick and Wilson (1992a,b) formulate these tensors in terms of integrals for monoclinic materials with the symmetry plane at $x_3 = 0$. Explicit expressions are then deduced for the special case of orthotropic and cubic materials.

In this paper, we obtain the explicit expressions of the matrices $\boldsymbol{L}(v)$, $\boldsymbol{S}(v)$ and $\boldsymbol{H}(v)$ for general anisotropic materials. The approach developed in our recent work for elastostatic problems (Liou and Sung, submitted for publication) is extended to the present derivations for the matrices $\boldsymbol{L}(v)$, $\boldsymbol{S}(v)$ and $\boldsymbol{H}(v)$. In previous work (Liou and Sung, submitted for publication), the Barnett–Lothe tensors $\boldsymbol{L}$, $\boldsymbol{S}$ and $\boldsymbol{H}$ were expressed in terms of elastic stiffness for general anisotropic materials. Similarly, all elements of matrices $\boldsymbol{L}(v)$, $\boldsymbol{S}(v)$ and $\boldsymbol{H}(v)$ are also expressed in terms of elastic stiffness and results for elastosatics (Liou and Sung, submitted for publication) may be recovered when $v = 0$. Explicit expressions of the matrices $\boldsymbol{L}(v)$, $\boldsymbol{S}(v)$ and $\boldsymbol{H}(v)$ derived for general anisotropic materials are then specialized to the cases of monoclinic materials with the plane of symmetry at $x_3 = 0$, $x_1 = 0$, and $x_2 = 0$. In particular, the results for the monoclinic materials with symmetry plane at $x_3 = 0$ remain valid for the degenerate cases when repeated eigenvalues occur. Moreover, our results of $\boldsymbol{L}(v)$, $\boldsymbol{S}(v)$ and $\boldsymbol{H}(v)$ for orthotropic materials may recover to those previously presented by Dongye and Ting (1989).

Below is the plan of our work. In Section 2, the Stroh formalism for the steady state motions of anisotropic elastic solids is outlined. In Section 3 the approach used by Liou and Sung (submitted for publication) was extended to the constructions of the eigenvectors for the steady state problems for anisotropic materials. With eigenvectors constructed in Section 3, the explicit expressions of the matrices $\boldsymbol{L}(v)$, $\boldsymbol{S}(v)$ and $\boldsymbol{H}(v)$ are then derived in Section 4. In Sections 5–7, results for materials with symmetry plane at $x_3 = 0$, $x_1 = 0$, and $x_2 = 0$ are deduced. In Section 8, our results were validated by the special case of orthotropic materials which were obtained by Dongye and Ting (1989) and finally in Section 9 we conclude our work.

2. The Stroh formalism

Consider a linear elastic body in a steady state motion in the x_1-direction with a constant speed $v > 0$. The governing equation for the displacement $\boldsymbol{u} = [u_1, u_2, u_3]^{\mathrm{T}}$ for the two-dimensional deformations for which u_i $(i = 1, 2, 3)$ are independent of x_3 is

$$(\boldsymbol{Q} - \rho v^2 \boldsymbol{I}) \frac{\partial^2 \boldsymbol{u}}{\partial x_1 \, \partial x_1} + (\boldsymbol{R} + \boldsymbol{R}^{\mathrm{T}}) \frac{\partial^2 \boldsymbol{u}}{\partial x_1 \, \partial x_2} + \boldsymbol{T} \frac{\partial^2 \boldsymbol{u}}{\partial x_2 \, \partial x_2} = \boldsymbol{0}, \tag{2.1}$$

where ρ is the mass density, $\boldsymbol{I}$ is a 3×3 unit real matrix, the superscript T stands for the transpose and

$$\boldsymbol{Q} = \begin{bmatrix} c_{11} & c_{16} & c_{15} \\ c_{16} & c_{66} & c_{56} \\ c_{15} & c_{56} & c_{55} \end{bmatrix}, \quad \boldsymbol{R} = \begin{bmatrix} c_{16} & c_{12} & c_{14} \\ c_{66} & c_{26} & c_{46} \\ c_{56} & c_{25} & c_{45} \end{bmatrix}, \quad \boldsymbol{T} = \begin{bmatrix} c_{66} & c_{26} & c_{46} \\ c_{26} & c_{22} & c_{24} \\ c_{46} & c_{24} & c_{44} \end{bmatrix}. \tag{2.2}$$

Here the contracted notations of the elastic stiffness c_{ijks} are used to express all the elements of $\boldsymbol{Q}$, $\boldsymbol{R}$ and $\boldsymbol{T}$ as shown above. Note that both $\boldsymbol{Q}$ and $\boldsymbol{T}$ are symmetric and positive definite. In what follows, only the subsonic problems are considered (Ting, 1996). Therefore, the general solution to Eq. (2.1) can be expressed as follows:

$$\boldsymbol{u} = 2\,\mathrm{Re}\{\boldsymbol{A}\boldsymbol{f}(z)\}, \tag{2.3}$$

where

$$\boldsymbol{A} = [\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3], \tag{2.4}$$

$$\boldsymbol{f}(z) = [f_1(z_1), f_2(z_2), f_3(z_3)]^{\mathrm{T}}, \tag{2.5}$$

and $z_k = x_1 - vt + p_k x_2$. Unknown complex number p_k and constant vector $\boldsymbol{a}_k$ are determined by the eigenrelation

$$\boldsymbol{U}\boldsymbol{a}_k = \boldsymbol{0}, \quad (k = 1, 2, 3), \tag{2.6}$$

where

$$\boldsymbol{U} = [\boldsymbol{Q} - \rho v^2 \boldsymbol{I} + p_k(\boldsymbol{R} + \boldsymbol{R}^{\mathrm{T}}) + p_k^2 \boldsymbol{T}]. \tag{2.7}$$

Introducing the stress function vector $\boldsymbol{\varphi} = [\varphi_1, \varphi_2, \varphi_3]^{\mathrm{T}}$, where

$$\boldsymbol{\varphi} = 2\,\mathrm{Re}\{\boldsymbol{B}\boldsymbol{f}(z)\}, \tag{2.8}$$

the stress components indicated below can be evaluated simply by

$$\boldsymbol{t}_1 = [\sigma_{11}, \sigma_{12}, \sigma_{13}]^{\mathrm{T}} = -\frac{\partial \boldsymbol{\varphi}}{\partial x_2} - \rho v \frac{\partial \boldsymbol{u}}{\partial t}, \quad \boldsymbol{t}_2 = [\sigma_{21}, \sigma_{22}, \sigma_{23}]^{\mathrm{T}} = \frac{\partial \boldsymbol{\varphi}}{\partial x_1}. \tag{2.9}$$

Note that the column vectors of matrix $\boldsymbol{B} = [\boldsymbol{b}_1, \boldsymbol{b}_2, \boldsymbol{b}_3]$ appearing in Eq. (2.8) are related to the column vectors of matrix $\boldsymbol{A} = [\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3]$ in the following form

$$\boldsymbol{b}_k = (\boldsymbol{R}^{\mathrm{T}} + p_k \boldsymbol{T})\boldsymbol{a}_k, \quad (k = 1, 2, 3). \tag{2.10}$$

Moreover, matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ defined above satisfy the orthogonality relations from which the three real matrices $\boldsymbol{L}(v)$, $\boldsymbol{S}(v)$ and $\boldsymbol{H}(v)$ may be defined as (Ting, 1996)

$$\boldsymbol{L}(v) = -2\mathrm{i}\boldsymbol{B}\boldsymbol{B}^{\mathrm{T}}, \quad \boldsymbol{S}(v) = \mathrm{i}(2\boldsymbol{A}\boldsymbol{B}^{\mathrm{T}} - \boldsymbol{I}), \quad \boldsymbol{H}(v) = 2\mathrm{i}\boldsymbol{A}\boldsymbol{A}^{\mathrm{T}}, \tag{2.11}$$

where $\mathrm{i}^2 = -1$. Matrices $\boldsymbol{L}(v)$ and $\boldsymbol{H}(v)$ are symmetric and positive definite. The defined three matrices $\boldsymbol{L}(v)$, $\boldsymbol{S}(v)$ and $\boldsymbol{H}(v)$ are not independent. They are related by

$$\boldsymbol{H}(v)\boldsymbol{L}(v) - \boldsymbol{S}(v)\boldsymbol{S}(v) = \boldsymbol{I}, \quad \boldsymbol{S}(v)\boldsymbol{H}(v) + \boldsymbol{H}(v)\boldsymbol{S}(v)^{\mathrm{T}} = \boldsymbol{0}, \quad \boldsymbol{L}(v)\boldsymbol{S}(v) + \boldsymbol{S}(v)^{\mathrm{T}}\boldsymbol{L}(v) = \boldsymbol{0}. \tag{2.12a, b, c}$$

3. The explicit expressions of A and B for anisotropic material

In this section the matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ are constructed in terms of the elastic stiffness for general anisotropic materials. For general anisotropic material, the matrix $\boldsymbol{U} = [\boldsymbol{Q} - \rho v^2 \boldsymbol{I} + p_k(\boldsymbol{R} + \boldsymbol{R}^{\mathrm{T}}) + p_k^2 \boldsymbol{T}]$, defined in Eq. (2.7) is

$$\boldsymbol{U} = \begin{bmatrix} (c_{66}p_k^2 + 2c_{16}p_k + c_{11}^*) & (c_{26}p_k^2 + (c_{12} + c_{66})p_k + c_{16}) & (c_{46}p_k^2 + (c_{14} + c_{56})p_k + c_{15}) \\ (c_{26}p_k^2 + (c_{12} + c_{66})p_k + c_{16}) & (c_{22}p_k^2 + 2c_{26}p_k + c_{66}^*) & (c_{24}p_k^2 + (c_{25} + c_{46})p_k + c_{56}) \\ (c_{46}p_k^2 + (c_{14} + c_{56})p_k + c_{15}) & (c_{24}p_k^2 + (c_{25} + c_{46})p_k + c_{56}) & (c_{44}p_k^2 + 2c_{45}p_k + c_{55}^*) \end{bmatrix}, \tag{3.1}$$

where $c_{kk}^* = c_{kk} - \rho v^2$, $(k = 1, 5, 6)$. To determine the eigenvectors $\boldsymbol{a}_k = [a_{1k}, a_{2k}, a_{3k}]^{\mathrm{T}}$, $(k = 1, 2, 3)$, from the eigenrelation defined in Eq. (2.10), it would be easier to construct the eigenvectors $\boldsymbol{b}_k = [b_{1k}, b_{2k}, b_{3k}]^{\mathrm{T}}$, $(k = 1, 2, 3)$ first. This is due to the fact that the condition of $\sigma_{12} = \sigma_{21}$ (from Eqs. (2.8) and (2.9)) implying that b_{1k}, b_{2k} and a_{2k} are actually related to each other by

$$b_{1k} + p_k b_{2k} - \rho v^2 a_{2k} = 0, \quad (k = 1, 2, 3). \tag{3.2}$$

Employing these conditions, the eigenvectors $\boldsymbol{b}_k = [b_{1k}, b_{2k}, b_{3k}]^{\mathrm{T}}$, $(k = 1, 2, 3)$ may be assumed in the following form

$$\boldsymbol{B} = [\boldsymbol{b}_1, \boldsymbol{b}_2, \boldsymbol{b}_3] = \tilde{\boldsymbol{B}} + \begin{bmatrix} \rho v^2 a_{21}(p_1) & \rho v^2 a_{22}(p_2) & \rho v^2 a_{23}(p_3) \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{3.3}$$

where

$$\tilde{\boldsymbol{B}} = [\tilde{\boldsymbol{b}}_1, \tilde{\boldsymbol{b}}_2, \tilde{\boldsymbol{b}}_3] = \begin{bmatrix} -p_1 & -p_2 & -p_3\lambda \\ 1 & 1 & \lambda \\ -\theta_1 & -\theta_2 & 1 \end{bmatrix} \tag{3.4}$$

and parameters θ_1, θ_2 and λ in matrix $\tilde{\boldsymbol{B}}$ are unknowns to be determined. Once the eigenvectors $\boldsymbol{b}_k=[b_{1k},b_{2k},b_{3k}]^{\mathrm{T}}$, $(k=1,2,3)$ were constructed above, the eigenvectors $\boldsymbol{a}_k=[a_{1k},a_{2k},a_{3k}]^{\mathrm{T}}$, $(k=1,2,3)$ may be determined through the relation in Eq. (2.10), i.e.,

$$\boldsymbol{a}_k=\left\{\boldsymbol{R}^{\mathrm{T}}+p_k\boldsymbol{T}-\rho v^2\begin{bmatrix}0&1&0\\0&0&0\\0&0&0\end{bmatrix}\right\}^{-1}\tilde{\boldsymbol{b}}_k, \tag{3.5}$$

where the use of Eq. (3.3) is made. Substitution of Eq. (3.4) into Eq. (3.5) and with some algebraic computations, it can be shown that the elements of $\boldsymbol{a}_k$ are as follows:

$$\begin{aligned}
a_{1k}(p_k)&=(\kappa_{11}(p_k)\theta_k(p_k)+\kappa_{10}(p_k))/\Delta(p_k),\quad (k=1,2),\\
a_{2k}(p_k)&=(\kappa_{21}(p_k)\theta_k(p_k)+\kappa_{20}(p_k))/\Delta(p_k),\quad (k=1,2),\\
a_{3k}(p_k)&=(\kappa_{31}(p_k)\theta_k(p_k)+\kappa_{30}(p_k))/\Delta(p_k),\quad (k=1,2),\\
a_{13}(p_3)&=(-\kappa_{11}(p_3)+\kappa_{10}(p_3)\lambda(p_3))/\Delta(p_3),\\
a_{23}(p_3)&=(-\kappa_{21}(p_3)+\kappa_{20}(p_3)\lambda(p_3))/\Delta(p_3),\\
a_{33}(p_3)&=(-\kappa_{31}(p_3)+\kappa_{30}(p_3)\lambda(p_3))/\Delta(p_3),
\end{aligned} \tag{3.6}$$

where

$$\begin{aligned}
\Delta(p_k)&=\Delta_3p_k^3+\Delta_2p_k^2+\Delta_1p_k+\Delta_0,\\
\Delta_3&=c_{66}c_{22}c_{44}+2c_{26}c_{24}c_{46}-c_{22}c_{46}^2-c_{44}c_{26}^2-c_{66}c_{24}^2,\\
\Delta_2&=c_{66}c_{22}c_{45}-c_{22}c_{56}c_{46}-c_{22}c_{46}c_{14}-c_{66}c_{25}c_{24}+c_{12}c_{24}c_{46}-c_{16}c_{24}^2\\
&\quad+c_{16}c_{22}c_{44}-c_{12}c_{26}c_{44}+c_{26}c_{25}c_{46}+c_{26}c_{24}c_{56}+c_{26}c_{24}c_{14}-c_{45}c_{26}^2\\
&\quad+c_{46}c_{24}(c_{66}^*-c_{66})-c_{44}c_{26}(c_{66}^*-c_{66}),\\
\Delta_1&=c_{16}c_{26}c_{44}-c_{22}c_{56}c_{14}-c_{12}c_{26}c_{45}-c_{16}c_{24}c_{46}+c_{16}c_{22}c_{45}+c_{14}c_{66}^*c_{24}\\
&\quad+c_{26}c_{25}c_{14}-c_{16}c_{25}c_{24}-c_{26}c_{46}c_{14}+c_{12}c_{24}c_{56}+c_{12}c_{46}^2-c_{44}c_{66}^*c_{12}\\
&\quad+c_{46}c_{25}(c_{66}^*-c_{66})-c_{45}c_{26}(c_{66}^*-c_{66}),\\
\Delta_0&=c_{14}c_{66}^*c_{25}+c_{16}c_{26}c_{45}+c_{12}c_{46}c_{56}-c_{16}c_{25}c_{46}-c_{14}c_{26}c_{56}-c_{12}c_{66}^*c_{45},
\end{aligned} \tag{3.7}$$

and

$$\begin{aligned}
\kappa_{11}(p_k)&=(c_{22}c_{46}-c_{26}c_{24})p_k^2+(c_{26}c_{46}+c_{22}c_{56}-c_{66}^*c_{24}-c_{26}c_{25})p_k+(c_{26}c_{56}-c_{66}^*c_{25}),\\
\kappa_{10}(p_k)&=(c_{24}^2-c_{22}c_{44})p_k^3+(2c_{24}c_{46}-2c_{26}c_{44}+c_{25}c_{24}-c_{22}c_{45})p_k^2\\
&\quad+(c_{25}c_{46}+c_{24}c_{56}-c_{66}^*c_{44}+c_{46}^2-2c_{26}c_{45})p_k+(c_{46}c_{56}-c_{66}^*c_{45}),\\
\kappa_{21}(p_k)&=(c_{66}c_{24}-c_{26}c_{46})p_k^2+(c_{16}c_{24}+c_{66}c_{25}-c_{26}c_{56}-c_{12}c_{46})p_k+(c_{16}c_{25}-c_{56}c_{12}),\\
\kappa_{20}(p_k)&=(c_{26}c_{44}-c_{24}c_{46})p_k^3+(c_{66}c_{44}+c_{26}c_{45}+c_{12}c_{44}-c_{24}c_{14}-c_{25}c_{46}-c_{46}^2)p_k^2\\
&\quad+(c_{66}c_{45}+c_{16}c_{44}+c_{12}c_{45}-c_{46}c_{14}-c_{46}c_{56}-c_{25}c_{14})p_k+(c_{16}c_{45}-c_{56}c_{14}),\\
\kappa_{31}(p_k)&=(c_{26}^2-c_{66}c_{22})p_k^2+[c_{26}c_{12}-c_{16}c_{22}+c_{26}(c_{66}^*-c_{66})]p_k+(c_{66}^*c_{12}-c_{16}c_{26}),\\
\kappa_{30}(p_k)&=(c_{22}c_{46}-c_{26}c_{24})p_k^3+(c_{22}c_{14}-c_{12}c_{24}+c_{26}c_{46}-c_{66}c_{24})p_k^2\\
&\quad+[2c_{26}c_{14}-c_{12}c_{46}-c_{16}c_{24}+c_{46}(c_{66}^*-c_{66})]p_k+(c_{66}^*c_{14}-c_{16}c_{46}),\ (k=1,2,3).
\end{aligned} \tag{3.8}$$

Now there remains the task to determine the unknown parameters θ_1, θ_2 and λ appearing in the eigenvectors $\boldsymbol{a}_k$ or $\boldsymbol{b}_k$. These unknown parameters θ_k $(k=1,2)$ and λ may be determined by the conditions that the three eigenvectors $\boldsymbol{a}_k$ constructed above have to satisfy the eigenrelation in Eq. (2.6). Applying these conditions to eigenvectors $\boldsymbol{a}_k$ and with lengthy algebraic manipulations (substituting Eq. (3.5) into Eq. (2.6)), a vector equation is arrived for θ_k, as

$$\boldsymbol{q}\equiv\boldsymbol{U}\boldsymbol{a}_k=[q_1(\theta_k),0,q_3(\theta_k)]^{\mathrm{T}}=\boldsymbol{0},\quad (k=1,2), \tag{3.9}$$

where

$$q_1(\theta_k) = -[\eta(p_k)\theta(p_k) + \zeta(p_k)]/\Delta(p_k), \quad (k = 1, 2), \tag{3.10a}$$
$$q_3(\theta_k) = -[\xi(p_k)\theta(p_k) + \beta(p_k)]/\Delta(p_k), \quad (k = 1, 2), \tag{3.10b}$$

and another vector equation is also arrived for λ as

$$\boldsymbol{q} \equiv \boldsymbol{U}\boldsymbol{a}_3 = [q_1(\lambda), 0, q_3(\lambda)]^{\mathrm{T}} = \boldsymbol{0}, \tag{3.11}$$

where

$$q_1(\lambda) = [\eta(p_3) - \zeta(p_3)\lambda(p_3)]/\Delta(p_3), \tag{3.12a}$$
$$q_3(\lambda) = [\xi(p_3) - \beta(p_3)\lambda(p_3)]/\Delta(p_3). \tag{3.12b}$$

In Eqs. (3.10) and (3.12) the expressions for $\xi(p_k)$, $\beta\ (p_k)$, $\eta(p_k)$, $\zeta(p_k)$, $(k = 1, 2, 3)$ are listed in Appendix A. Note that in the vector equation for the determination of θ_k there are two conditions available, i.e., $q_1(\theta_k) = 0$ and $q_3(\theta_k) = 0$, as shown in Eq. (3.9). Enforcing first the condition $q_3(\theta_k) = 0$, then θ_k can be determined as

$$\theta_k(p_k) = -\frac{\beta(p_k)}{\xi(p_k)}, \quad (k = 1, 2). \tag{3.13}$$

It can be verified, with the identity

$$\zeta(p_k)\xi(p_k) - \eta(p_k)\beta(p_k) = \Delta(p_k)|\boldsymbol{U}(p_k)|, \quad (k = 1,\ 2,\ 3), \tag{3.14}$$

that the values of θ_k determined by the condition $q_3(\theta_k) = 0$ automatically satisfy the other condition in Eq. (3.10a), i.e., $q_1(\theta_k) = 0$. Therefore only one condition in the vector equation for θ_k needed to be considered. Similarly, in the vector equation for the determination of λ there are two conditions available, i.e., $q_1(\lambda) = 0$ and $q_3(\lambda) = 0$, as shown in Eq. (3.11). Enforcing first now the condition $q_1(\lambda) = 0$ in Eq. (3.12a), λ can be determined as

$$\lambda(p_3) = \frac{\eta(p_3)}{\zeta(p_3)}. \tag{3.15}$$

It can also be verified, with the identity shown in Eq. (3.14), that the values of λ determined by the condition $q_1(\lambda) = 0$ automatically satisfy the other condition in Eq. (3.12b), i.e., $q_3(\lambda) = 0$. So far the unknown parameters are completely determined and are completely expressed in terms of the elastic stiffness. Hence the matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ are also expressed all in terms of the elastic stiffness.

4. Explicit expressions of $\boldsymbol{L}(v)$, $\boldsymbol{S}(v)$ and $\boldsymbol{H}(v)$ for anisotropic material

With the matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ constructed in previously section, the three real matrices $\boldsymbol{L}(v)$, $\boldsymbol{S}(v)$ and $\boldsymbol{H}(v)$ defined in Eq. (2.11) can be determined without difficulty. It is known that the surface impedance matrix $\boldsymbol{M}$ defined as

$$\boldsymbol{Y} = \boldsymbol{M}^{-1} \equiv \mathrm{i}\boldsymbol{A}\boldsymbol{B}^{-1} = \boldsymbol{L}^{-1}(v) - \mathrm{i}\boldsymbol{S}(v)\boldsymbol{L}^{-1}(v) \tag{4.1}$$

is related to the matrices $\boldsymbol{L}(v)$ and $\boldsymbol{S}(v)$. Therefore we may construct matrix $\boldsymbol{Y}$ first by multiplications of $\boldsymbol{A}$ and $\boldsymbol{B}^{-1}$, and then take the real and imaginary part of $\boldsymbol{Y}$, respectively, matrices $\boldsymbol{L}^{-1}(v)$ and $\boldsymbol{S}(v)\boldsymbol{L}^{-1}(v)$ may be obtained. Using the procedure just described, we obtain

$$\boldsymbol{L}^{-1}(v) = \begin{bmatrix} Y_{11} & Y_{12} & Y_{13} \\ Y_{12} & Y_{22} & Y_{23} \\ Y_{13} & Y_{23} & Y_{33} \end{bmatrix}, \quad \boldsymbol{S}(v)\boldsymbol{L}^{-1}(v) = \begin{bmatrix} 0 & -\hat{Y}_{12} & -\hat{Y}_{13} \\ \hat{Y}_{12} & 0 & -\hat{Y}_{23} \\ \hat{Y}_{13} & \hat{Y}_{23} & 0 \end{bmatrix}, \tag{4.2}$$

where

$$
\begin{aligned}
&Y_{11} = \mathrm{i}[\Theta_0 + \lambda\Theta_1 - a_{13}\Theta]/D_B, \quad Y_{22} = \mathrm{i}[\lambda p_3\Theta_2 + a_{23}\Gamma + \Gamma_2]/D_B,\\
&Y_{33} = \mathrm{i}[\lambda p_3\Theta_3 - \lambda\Gamma_3 + a_{33} + \rho v^2(\lambda a_{21}\Theta_3 - \lambda a_{31}\Gamma_0 - a_{23}\Theta_3 + a_{33}\Gamma_0)]/D_B,\\
&Y_{12} = -\mathrm{Im}\{[\lambda p_3\Theta_1 + a_{13}\Gamma + \Gamma_1 - \rho v^2(a_{22}\Theta_0 - a_{12}\Gamma_0 + a_{23}\Theta_1 - a_{13}\Theta_2)]/D_B\},\\
&\hat{Y}_{12} = \mathrm{Re}\{[\lambda p_3\Theta_1 + a_{13}\Gamma + \Gamma_1 - \rho v^2(a_{22}\Theta_0 - a_{12}\Gamma_0 + a_{23}\Theta_1 - a_{13}\Theta_2)]/D_B\},\\
&Y_{13} = -\mathrm{Im}\{[\lambda p_3\Theta_0 - \lambda\Gamma_1 + a_{13} - \rho v^2(\lambda a_{12}\Gamma_0 - \lambda a_{22}\Theta_0 + a_{23}\Theta_0 - a_{13}\Gamma_0)]/D_B\},\\
&\hat{Y}_{13} = \mathrm{Re}\{[\lambda p_3\Theta_0 - \lambda\Gamma_1 + a_{13} - \rho v^2(\lambda a_{12}\Gamma_0 - \lambda a_{22}\Theta_0 + a_{23}\Theta_0 - a_{13}\Gamma_0)]/D_B\},\\
&Y_{23} = -\mathrm{Im}\{[\lambda p_3\Gamma_0 - \lambda\Gamma_2 + a_{23}]/D_B\}, \quad \hat{Y}_{23} = \mathrm{Re}\{[\lambda p_3\Gamma_0 - \lambda\Gamma_2 + a_{23}]/D_B\},
\end{aligned}
\tag{4.3}
$$

where $D_B = 1 + \lambda(\Gamma + p_3\Theta) + \rho v^2(\Gamma_0 + \lambda\Theta_2 - a_{23}\Theta)$ and

$$
\begin{aligned}
\Theta_0 &= (c_{22}c_{46} - c_{26}c_{24})\Lambda_2 + (c_{26}c_{46} + c_{22}c_{56} - c^*_{66}c_{24} - c_{26}c_{25})\Lambda_1 + (c_{26}c_{56} - c^*_{66}c_{25})\Lambda_0\\
&\quad + (c_{24}^2 - c_{22}c_{44})\Omega_3 + (2c_{24}c_{46} - 2c_{26}c_{44} + c_{25}c_{24} - c_{22}c_{45})\Omega_2\\
&\quad + (c_{25}c_{46} + c_{24}c_{56} - c^*_{66}c_{44} + c_{46}^2 - 2c_{26}c_{45})\Omega_1 + (c_{46}c_{56} - c^*_{66}c_{45})\Omega_0,\\
\Theta_1 &= \theta_1\theta_2[(c_{22}c_{46} - c_{26}c_{24})\Omega_2 + (c_{26}c_{46} + c_{22}c_{56} - c^*_{66}c_{24} - c_{26}c_{25})\Omega_1 + (c_{26}c_{56} - c^*_{66}c_{25})\Omega_0]\\
&\quad + ((c_{24}^2 - c_{22}c_{44})\Pi_3 + (2c_{24}c_{46} - 2c_{26}c_{44} + c_{25}c_{24} - c_{22}c_{45})\Pi_2\\
&\quad + (c_{25}c_{46} + c_{24}c_{56} - c^*_{66}c_{44} + c_{46}^2 - 2c_{26}c_{45})\Pi_1 + (c_{46}c_{56} - c^*_{66}c_{45})\Pi_0,\\
\Theta_2 &= \theta_1\theta_2[(c_{66}c_{24} - c_{26}c_{46})\Omega_2 + (c_{16}c_{24} + c_{66}c_{25} - c_{26}c_{56} - c_{12}c_{46})\Omega_1\\
&\quad + (c_{16}c_{25} - c_{56}c_{12})\Omega_0] + (c_{26}c_{44} - c_{24}c_{46})\Pi_3 + (c_{16}c_{45} - c_{56}c_{14})\Pi_0\\
&\quad + (c_{66}c_{44} + c_{26}c_{45} + c_{12}c_{44} - c_{24}c_{14} - c_{25}c_{46} - c_{46}^2)\Pi_2\\
&\quad + (c_{66}c_{45} + c_{16}c_{44} + c_{12}c_{45} - c_{46}c_{14} - c_{46}c_{56} - c_{25}c_{14})\Pi_1,\\
\Theta_3 &= (c_{26}^2 - c_{66}c_{22})\Lambda_2 + [c_{26}c_{12} - c_{16}c_{22} + c_{26}(c^*_{66} - c_{66})]\Lambda_1 + (c^*_{66}c_{12} - c_{16}c_{26})\Lambda_0\\
&\quad + (c_{22}c_{46} - c_{26}c_{24})\Omega_3 + (c_{22}c_{14} - c_{12}c_{24} + c_{26}c_{46} - c_{66}c_{24})\Omega_2\\
&\quad + [2c_{26}c_{14} - c_{12}c_{46} - c_{16}c_{24} + c_{46}(c^*_{66} - c_{66})]\Omega_1 + (c^*_{66}c_{14} - c_{16}c_{46})\Omega_0,\\
\Gamma_0 &= (c_{66}c_{24} - c_{26}c_{46})\Lambda_2 + (c_{16}c_{24} + c_{66}c_{25} - c_{26}c_{56} - c_{12}c_{46})\Lambda_1 + (c_{16}c_{25} - c_{56}c_{12})\Lambda_0\\
&\quad + (c_{26}c_{44} - c_{24}c_{46})\Omega_3 + (c_{66}c_{44} + c_{26}c_{45} + c_{12}c_{44} - c_{24}c_{14} - c_{25}c_{46} - c_{46}^2)\Omega_2\\
&\quad + (c_{66}c_{45} + c_{16}c_{44} + c_{12}c_{45} - c_{46}c_{14} - c_{46}c_{56} - c_{25}c_{14})\Omega_1 + (c_{16}c_{45} - c_{56}c_{14})\Omega_0,\\
\Gamma_1 &= (c_{22}c_{46} - c_{26}c_{24})p_1p_2\Lambda_1 + (c_{26}c_{46} + c_{22}c_{56} - c^*_{66}c_{24} - c_{26}c_{25})p_1p_2\Lambda_0 + (c_{26}c_{56} - c^*_{66}c_{25})\Lambda_4\\
&\quad + (c_{24}^2 - c_{22}c_{44})p_1p_2\Omega_2 + (2c_{24}c_{46} - 2c_{26}c_{44} + c_{25}c_{24} - c_{22}c_{45})p_1p_2\Omega_1\\
&\quad + (c_{25}c_{46} + c_{24}c_{56} - c^*_{66}c_{44} + c_{46}^2 - 2c_{26}c_{45})p_1p_2\Omega_0 + (c_{46}c_{56} - c^*_{66}c_{45})\Omega_4,\\
\Gamma_2 &= (c_{66}c_{24} - c_{26}c_{46})p_1p_2\Lambda_1 + (c_{16}c_{24} + c_{66}c_{25} - c_{26}c_{56} - c_{12}c_{46})p_1p_2\Lambda_0 + (c_{16}c_{25} - c_{56}c_{12})\Lambda_4\\
&\quad + (c_{26}c_{44} - c_{24}c_{46})p_1p_2\Omega_2 + (c_{66}c_{44} + c_{26}c_{45} + c_{12}c_{44} - c_{24}c_{14} - c_{25}c_{46} - c_{46}^2)p_1p_2\Omega_1\\
&\quad + (c_{66}c_{45} + c_{16}c_{44} + c_{12}c_{45} - c_{46}c_{14} - c_{46}c_{56} - c_{25}c_{14})p_1p_2\Omega_0 + (c_{16}c_{45} - c_{56}c_{14})\Omega_4,\\
\Gamma_3 &= (c_{26}^2 - c_{66}c_{22})p_1p_2\Lambda_1 + [c_{26}c_{12} - c_{16}c_{22} + c_{26}(c^*_{66} - c_{66})]p_1p_2\Lambda_0 + (c^*_{66}c_{12} - c_{16}c_{26})\Lambda_4\\
&\quad + (c_{22}c_{46} - c_{26}c_{24})p_1p_2\Omega_2 + (c_{22}c_{14} - c_{12}c_{24} + c_{26}c_{46} - c_{66}c_{24})p_1p_2\Omega_1\\
&\quad + [2c_{26}c_{14} - c_{12}c_{46} - c_{16}c_{24} + c_{46}(c^*_{66} - c_{66})]p_1p_2\Omega_0 + (c^*_{66}c_{14} - c_{16}c_{46})\Omega_4,
\end{aligned}
\tag{4.4}
$$

and

$$\begin{aligned}
&\Theta=\frac{\theta_2-\theta_1}{p_2-p_1},\quad \Gamma=\frac{\theta_1p_2-\theta_2p_1}{p_2-p_1},\quad \Omega_0=\frac{(\frac{1}{\Delta_1}-\frac{1}{\Delta_2})}{(p_2-p_1)},\\
&\Omega_1=\frac{(\frac{p_1}{\Delta_1}-\frac{p_2}{\Delta_2})}{(p_2-p_1)},\quad \Omega_2=\frac{(\frac{p_1^2}{\Delta_1}-\frac{p_2^2}{\Delta_2})}{(p_2-p_1)},\quad \Omega_3=\frac{(\frac{p_1^3}{\Delta_1}-\frac{p_2^3}{\Delta_2})}{(p_2-p_1)},\quad \Omega_4=\frac{(\frac{p_2}{\Delta_1}-\frac{p_1}{\Delta_2})}{(p_2-p_1)},\\
&\Lambda_0=\frac{(\frac{\theta_1}{\Delta_1}-\frac{\theta_2}{\Delta_2})}{(p_2-p_1)},\quad \Lambda_1=\frac{(\frac{\theta_1}{\Delta_1}p_1-\frac{\theta_2}{\Delta_2}p_2)}{(p_2-p_1)},\quad \Lambda_2=\frac{(\frac{\theta_1}{\Delta_1}p_1^2-\frac{\theta_2}{\Delta_2}p_2^2)}{(p_2-p_1)},\quad \Lambda_4=\frac{(\frac{\theta_1}{\Delta_1}p_2-\frac{\theta_2}{\Delta_2}p_1)}{(p_2-p_1)},\\
&\Pi_0=\frac{(\frac{\theta_2}{\Delta_1}-\frac{\theta_1}{\Delta_2})}{(p_2-p_1)},\quad \Pi_1=\frac{(\frac{\theta_2}{\Delta_1}p_1-\frac{\theta_1}{\Delta_2}p_2)}{(p_2-p_1)},\quad \Pi_2=\frac{(\frac{\theta_2}{\Delta_1}p_1^2-\frac{\theta_1}{\Delta_2}p_2^2)}{(p_2-p_1)},\quad \Pi_3=\frac{(\frac{\theta_2}{\Delta_1}p_1^3-\frac{\theta_1}{\Delta_2}p_2^3)}{(p_2-p_1)}
\end{aligned}\tag{4.5}$$

be computed by the same procedures as described in the paper by Liou and Sung (submitted for publication). The obtained explicit formulae for the elements of these matrices are the same as those three Barnett–Lothe tensors for elastostatics (Liou and Sung, submitted for publication). Therefore, the obtained explicit formulae for these matrices are not reproduced here. Instead, they are listed in Appendix B for readers' convenience. It is noted that although the explicit formulae for both elastodynamics and elastostatics are the same, the parameters appearing in the formulae for $\boldsymbol{L}(v)$, $\boldsymbol{S}(v)$ and $\boldsymbol{H}(v)$ as presented in Eqs. (4.3) and (4.4) will take different expressions which account for the dynamic effects, and the expressions for these parameters (Eqs. (4.3) and (4.4))will be reduced to those for elastostatics when $v=0$. In the following sections, results of $\boldsymbol{L}(v)$, $\boldsymbol{S}(v)$ and $\boldsymbol{H}(v)$ for materials with special symmetric planes will be presented. The formulae obtained for matrices $\boldsymbol{L}(v)$, $\boldsymbol{S}(v)$ and $\boldsymbol{H}(v)$ for each of a special material under consideration will again take the same form as those for elastostatics and the expressions for elastostatics may be recovered from those expressions for the elastodynamics by setting $v=0$.

5. Monoclinic materials with the symmetry plane at $x_3 = 0$

In this section the explicit expressions for matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ and the three matrices $\boldsymbol{L}(v)$, $\boldsymbol{S}(v)$ and $\boldsymbol{H}(v)$ for monoclinic materials with the symmetry plane at $x_3=0$ will be presented. For monoclinic materials with the symmetry plane at $x_3=0$, the elastic stiffness can be obtained from general anisotropic materials by letting

$$c_{14}=c_{15}=c_{24}=c_{25}=c_{46}=c_{56}=0.\tag{5.1}$$

The conditions of Eq. (5.1) imply that $\beta(p_1)=\beta(p_2)=\eta(p_3)=0$ and $\theta_1=\theta_2=\lambda=0$, so that the eigenvectors $\boldsymbol{a}_k$ and $\boldsymbol{b}_k$ may be written in a simpler form

$$\boldsymbol{a}_k=[a_{1k},a_{2k},0]^{\mathrm{T}},\quad (k=1,2),\quad \boldsymbol{a}_3=[0,0,a_{33}]^{\mathrm{T}},\tag{5.2}$$

$$\boldsymbol{b}_k=[-p_k+\rho v^2a_{2k},1,0]^{\mathrm{T}},\quad (k=1,2),\quad \boldsymbol{b}_3=[0,0,1]^{\mathrm{T}},\tag{5.3}$$

and the elements of matrices $\boldsymbol{A}$ and $\boldsymbol{B}$, which may be specialized directly from the general results obtained in Section 3, are expressed as

$$a_{1k}(p_k)=-(c_{22}p_k^2+2c_{26}p_k+c_{66}^*)/\Delta(p_k),\tag{5.4a}$$

$$a_{2k}(p_k)=(c_{26}p_k^2+(c_{12}+c_{66})p_k+c_{16})/\Delta(p_k),\quad (k=1,2),\tag{5.4b}$$

$$a_{33}(p_3)=(c_{45}+p_3c_{44})^{-1},\tag{5.4c}$$

where

$$\Delta(p_k)=(c_{22}c_{66}-c_{26}^2)p_k^2+[c_{16}c_{22}-c_{12}c_{26}-c_{26}(c_{66}^*-c_{66})]p_k+(c_{16}c_{26}-c_{12}c_{66}^*).\tag{5.5}$$

The procedures of the computations of the matrices $\boldsymbol{L}^{-1}(v)$, $\boldsymbol{S}(v)\boldsymbol{L}^{-1}(v)$, $\boldsymbol{L}(v)$, $\boldsymbol{S}(v)$ and $\boldsymbol{H}(v)$ follow those described in Section 4. In the derivations of these matrices, some properties of the eigenvalues for monoclinic materials with the symmetry plane at $x_3=0$ described below are needed. First note that for general anisotropic materials the eigenvalues are determined by the following sextic equation

$$\delta=|\boldsymbol{U}|=0,\tag{5.6}$$

where $\boldsymbol{U}$ is defined by Eq. (3.1). This sextic equation uncouples into a quadratic equation

$$c_{44}p^2 + 2c_{45}p + c^*_{55} = 0 \tag{5.7a}$$

and a quartic equation

$$\begin{aligned}(c_{22}c_{66} - c^2_{26})p^4 - 2(c_{12}c_{26} - c_{16}c_{22})p^3 + [2(c_{16}c_{26} - c_{12}c_{66}) + c^*_{11}c_{22} - c^2_{12} + c_{66}(c^*_{66} - c_{66})]p^2 \\ - 2[c_{12}c_{16} - c^*_{11}c_{26} - c_{16}(c^*_{66} - c_{66})]p + (c^*_{11}c^*_{66} - c^2_{16}) = 0,\end{aligned} \tag{5.7b}$$

for monoclinic materials with the symmetry plane at $x_3 = 0$. Therefore, the roots with positive imaginary parts of above two equations may be expressed as

$$p_1 = m_1 + \mathrm{i}n_1, \quad p_2 = m_2 + \mathrm{i}n_2, \quad p_3 = \frac{-c_{45} + \mathrm{i}(c_{44}c^*_{55} - c^2_{45})^{1/2}}{c_{44}} \tag{5.8}$$

in which m_k $(k = 1,2)$ are real and n_k $(k = 1,2)$ are real and positive. It is then easily verified that the element Y_{33} of the third row and third column of the matrix $\boldsymbol{L}^{-1}(v)$ may be simplified using the property of p_3 in Eq. (5.8). Furthermore the elements Y_{13} and Y_{23} of $\boldsymbol{L}^{-1}(v)$ and the elements $\hat{Y}_{13}$ and $\hat{Y}_{23}$ of $\boldsymbol{S}(v)\boldsymbol{L}^{-1}(v)$ are all found to vanish if the conditions in Eq. (5.1) is imposed. With these facts described above and with some algebraic calculations, it can be shown that the results of $\boldsymbol{L}^{-1}(v)$, $\boldsymbol{S}(v)\boldsymbol{L}^{-1}(v)$, $\boldsymbol{L}(v)$, $\boldsymbol{S}(v)$ and $\boldsymbol{H}(v)$ are

$$\begin{aligned}
&\boldsymbol{L}^{-1}(v) = \begin{bmatrix} Y_{11} & Y_{12} & 0 \\ Y_{12} & Y_{22} & 0 \\ 0 & 0 & Y_{33} \end{bmatrix}, \quad \boldsymbol{S}(v)\boldsymbol{L}^{-1}(v) = \begin{bmatrix} 0 & -\hat{Y}_{12} & 0 \\ \hat{Y}_{12} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \\
&\boldsymbol{L}(v) = \begin{bmatrix} \frac{Y_{22}}{Y_{11}Y_{22}-Y^2_{12}} & \frac{-Y_{12}}{Y_{11}Y_{22}-Y^2_{12}} & 0 \\ \frac{-Y_{12}}{Y_{11}Y_{22}-Y^2_{12}} & \frac{Y_{11}}{Y_{11}Y_{22}-Y^2_{12}} & 0 \\ 0 & 0 & \frac{1}{Y_{33}} \end{bmatrix}, \quad \boldsymbol{S}(v) = \frac{\hat{Y}_{12}}{Y_{11}Y_{22} - Y^2_{12}} \begin{bmatrix} Y_{12} & -Y_{11} & 0 \\ Y_{22} & -Y_{12} & 0 \\ 0 & 0 & 0 \end{bmatrix}, \\
&\boldsymbol{H}(v) = \frac{-1}{Y_{11}Y_{22} - Y^2_{12}} \begin{bmatrix} Y_{11}(Y^2_{12} - Y_{11}Y_{22} + \hat{Y}^2_{12}) & Y_{12}(Y^2_{12} - Y_{11}Y_{22} + \hat{Y}^2_{12}) & 0 \\ Y_{12}(Y^2_{12} - Y_{11}Y_{22} + \hat{Y}^2_{12}) & Y_{22}(Y^2_{12} - Y_{11}Y_{22} + \hat{Y}^2_{12}) & 0 \\ 0 & 0 & Y_{33}(Y^2_{12} - Y_{11}Y_{22}) \end{bmatrix},
\end{aligned} \tag{5.9}$$

where

$$\begin{aligned}
&Y_{11} = -\mathrm{i}[(c^*_{66}\Omega_0 + 2c_{26}\Omega_1 + c_{22}\Omega_2)/D_B], \\
&Y_{22} = \mathrm{i}\{[c_{16}\Omega_4 + (p_1p_2)((c_{12} + c_{66})\Omega_0 + c_{26}\Omega_1)]/D_B\}, \\
&Y_{12} = \mathrm{Im}\{[c^*_{66}\Omega_4 + (p_1p_2)(2c_{26}\Omega_0 + c_{22}\Omega_1) + \rho v^2\Omega_5]/D_B\}, \\
&\hat{Y}_{12} = -\mathrm{Re}\{[c^*_{66}\Omega_4 + (p_1p_2)(2c_{26}\Omega_0 + c_{22}\Omega_1) + \rho v^2\Omega_5]/D_B\}, \\
&Y_{33} = (c_{44}c^*_{55} - c^2_{45})^{-1/2},
\end{aligned} \tag{5.10}$$

where

$$
\begin{aligned}
D_B &= 1+\rho v^2\Gamma_0, \quad \Gamma_0 = c_{26}\Omega_2 + (c_{12}+c_{66})\Omega_1 + c_{16}\Omega_0,\\
\Omega_0 &= \frac{1}{\Delta(p_1)\Delta(p_2)}\{(c_{22}c_{66}-c_{26}^2)(p_1+p_2) + [c_{16}c_{22} - c_{12}c_{26} - c_{26}(c_{66}^* - c_{66})]\},\\
\Omega_1 &= \frac{1}{\Delta(p_1)\Delta(p_2)}[(c_{22}c_{66}-c_{26}^2)(p_1p_2) + (c_{12}c_{66}^* - c_{16}c_{26})],\\
\Omega_2 &= \frac{1}{\Delta(p_1)\Delta(p_2)}\{(c_{12}c_{66}^* - c_{16}c_{26})(p_1+p_2) + [c_{12}c_{26} - c_{16}c_{22} + c_{26}(c_{66}^* - c_{66})](p_1p_2)\},\\
\Omega_4 &= \frac{1}{\Delta(p_1)\Delta(p_2)}\{(c_{16}c_{26} - c_{12}c_{66}^*) + (c_{22}c_{66}-c_{26}^2)(p_1^2+p_2^2+p_1p_2)\\
&\qquad + [c_{16}c_{22} - c_{12}c_{26} - c_{26}(c_{66}^* - c_{66})](p_1+p_2)\},\\
\Omega_5 &= \frac{1}{\Delta(p_1)\Delta(p_2)}\{[c_{22}(c_{12}+c_{66}) - 2c_{26}c_{26}]p_1p_2 + (c_{22}c_{16} - c_{66}^*c_{26})(p_1+p_2) + [2c_{26}c_{16} - c_{66}^*(c_{12}+c_{66})]\}.
\end{aligned} \tag{5.11}
$$

Note that the explicit formulae (Eq. (5.9))for the elements of matrices $\boldsymbol{L}(v)$, $\boldsymbol{S}(v)$ and $\boldsymbol{H}(v)$ for monoclinic materials with the symmetry plane at $x_3 = 0$ are the same as those for elastosatics (Liou and Sung, submitted for publication). However, parameters expressed in Eqs. (5.10) and (5.11) contain the elastodynamic effects and they are reduced to those for elastostatics when $v = 0$ (Liou and Sung, submitted for publication). Note also that these parameters are expressed in terms of the elastic stiffness and eigenvalues. They are independent of the eigenvectors. Therefore matrices $\boldsymbol{L}(v)$, $\boldsymbol{S}(v)$ and $\boldsymbol{H}(v)$ obtained above are still valid for the degenerate problems when repeated eigenvalues occur.

6. Monoclinic materials with the symmetry plane at $x_1 = 0$

For monoclinic materials with the symmetry plane at $x_1 = 0$, the elastic stiffness can be obtained from general anisotropic materials by letting $c_{15} = c_{16} = c_{25} = c_{26} = c_{45} = c_{46} = 0$. With these vanishing elements, the sextic equation in Eq. (5.6) becomes

$$\delta = \delta_6 p^6 + \delta_4 p^4 + \delta_2 p^2 + \delta_0 = 0, \tag{6.1}$$

where

$$
\begin{aligned}
\delta_6 &= c_{66}c_{22}c_{44} - c_{66}c_{24}^2,\\
\delta_4 &= 2c_{12}c_{24}c_{14} + 2c_{12}c_{24}c_{56} + 2c_{66}c_{24}c_{14} - 2c_{12}c_{66}c_{44} - 2c_{14}c_{56}c_{22}\\
&\quad + c_{11}^*c_{22}c_{44} + c_{66}c_{22}c_{55}^* - c_{22}c_{14}^2 - c_{44}c_{12}^2 - c_{22}c_{56}^2 - c_{11}^*c_{24}^2 + c_{44}c_{66}(c_{66}^* - c_{66}),\\
\delta_2 &= c_{11}^*c_{22}c_{55}^* + c_{11}^*c_{66}^*c_{44} - 2c_{11}^*c_{56}c_{24} - 2c_{12}c_{66}c_{55}^* + 2c_{12}c_{56}c_{14} - c_{66}^*c_{14}^2\\
&\quad + 2c_{12}c_{56}^2 - c_{55}^*c_{12}^2 + c_{55}^*c_{66}(c_{66}^* - c_{66}) - c_{56}^2(c_{66}^* - c_{66}) - 2c_{14}c_{56}(c_{66}^* - c_{66}),\\
\delta_0 &= c_{11}^*c_{55}^*c_{66}^* - c_{11}^*c_{56}^2.
\end{aligned} \tag{6.2}
$$

The roots corresponding to Eq. (6.1) can be classified into two types (Wei and Ting, 1994). For the type I, the roots are

$$p_1 = \mathrm{i}n_1, \quad p_2 = \mathrm{i}n_2, \quad p_3 = \mathrm{i}n_3 \quad (n_1 > 0,\ n_2 > 0,\ n_3 > 0), \tag{6.3a}$$

which are all purely imaginary. As to type II, the roots are

$$p_1 = m_1 + \mathrm{i}n_1, \quad p_2 = -m_1 + \mathrm{i}n_1, \quad p_3 = \mathrm{i}n_3. \tag{6.3b}$$

Matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ for this material may be specialized again from the general results presented in Section 3. The parameters in Eq. (3.6) are simplified to

$$
\begin{aligned}
&\kappa_{11}(p_k) = (c_{22}c_{56} - c_{66}^*c_{24})p_k,\\
&\kappa_{10}(p_k) = (c_{24}^2 - c_{22}c_{44})p_k^3 + (c_{24}c_{56} - c_{66}^*c_{44})p_k,\\
&\kappa_{21}(p_k) = c_{66}c_{24}p_k^2 - c_{56}c_{12},\\
&\kappa_{20}(p_k) = (c_{66}c_{44} + c_{12}c_{44} - c_{24}c_{14})p_k^2 - c_{56}c_{14}, \quad (k = 1,2,3),\\
&\kappa_{31}(p_k) = -c_{66}c_{22}p_k^2 + c_{66}^*c_{12},\\
&\kappa_{30}(p_k) = (c_{22}c_{14} - c_{12}c_{24} - c_{66}c_{24})p_k^2 + c_{66}^*c_{14},\\
&\Delta(p_k) = (c_{66}c_{22}c_{44} - c_{66}c_{24}^2)p_k^3 + (c_{66}^*c_{24}c_{14} + c_{12}c_{24}c_{56} - c_{22}c_{56}c_{14} - c_{12}c_{66}^*c_{44})p_k.
\end{aligned} \tag{6.4}
$$

Similarly, the parameters $\theta_k(p_k)$, $(k = 1,2)$ and $\lambda(p_3)$ given by Eqs. (3.13) and (3.15), respectively, may also be simplified with new expressions for $\eta(p_3)$, $\zeta(p_3)$, $\xi(p_k)$, $\beta(p_k)$, $(k = 1,2)$ which are given in Appendix A. With matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ obtained above and with the properties of the eigenvalues in Eq. (6.3), matrices $\boldsymbol{L}^{-1}(v)$, $\boldsymbol{S}(v)\boldsymbol{L}^{-1}(v)$, $\boldsymbol{L}(v)$, $\boldsymbol{S}(v)$ and $\boldsymbol{H}(v)$ are obtained as

$$
\begin{aligned}
&\boldsymbol{L}^{-1}(v) = \begin{bmatrix} Y_{11} & 0 & 0\\ 0 & Y_{22} & Y_{23}\\ 0 & Y_{23} & Y_{33} \end{bmatrix}, \quad \boldsymbol{S}(v)\boldsymbol{L}^{-1}(v) = \begin{bmatrix} 0 & -\hat{Y}_{12} & -\hat{Y}_{13}\\ \hat{Y}_{12} & 0 & 0\\ \hat{Y}_{13} & 0 & 0 \end{bmatrix},\\
&\boldsymbol{L}(v) = \begin{bmatrix} \frac{1}{Y_{11}} & 0 & 0\\ 0 & \frac{Y_{33}}{Y_{22}Y_{33}-Y_{23}^2} & \frac{-Y_{23}}{Y_{22}Y_{33}-Y_{23}^2}\\ 0 & \frac{-Y_{23}}{Y_{22}Y_{33}-Y_{23}^2} & \frac{Y_{22}}{Y_{22}Y_{33}-Y_{23}^2} \end{bmatrix}, \quad \boldsymbol{S}(v) = \begin{bmatrix} 0 & \frac{\hat{Y}_{13}Y_{23}-\hat{Y}_{12}Y_{33}}{Y_{22}Y_{33}-Y_{23}^2} & \frac{\hat{Y}_{12}Y_{23}-\hat{Y}_{13}Y_{22}}{Y_{22}Y_{33}-Y_{23}^2}\\ \frac{\hat{Y}_{12}}{Y_{11}} & 0 & 0\\ \frac{\hat{Y}_{13}}{Y_{11}} & 0 & 0 \end{bmatrix},\\
&\boldsymbol{H}(v) = \begin{bmatrix} Y_{11} - \frac{\hat{Y}_{12}^2Y_{33}-2\hat{Y}_{12}\hat{Y}_{13}Y_{23}+\hat{Y}_{13}^2Y_{22}}{Y_{22}Y_{33}-Y_{23}^2} & 0 & 0\\ 0 & Y_{22} - \frac{\hat{Y}_{12}^2}{Y_{11}} & Y_{23} - \frac{\hat{Y}_{12}\hat{Y}_{13}}{Y_{11}}\\ 0 & Y_{23} - \frac{\hat{Y}_{12}\hat{Y}_{13}}{Y_{11}} & Y_{33} - \frac{\hat{Y}_{13}^2}{Y_{11}} \end{bmatrix},
\end{aligned} \tag{6.5}
$$

where

$$
\begin{aligned}
&Y_{11} = \mathrm{i}[\Theta_0 + \lambda\Theta_1 - a_{13}\Theta]/D_B, \quad Y_{22} = \mathrm{i}[\lambda p_3\Theta_2 + a_{23}\Gamma + \Gamma_2]/D_B,\\
&Y_{33} = \mathrm{i}[\lambda p_3\Theta_3 - \lambda\Gamma_3 + a_{33} + \rho v^2(\lambda a_{21}\Theta_3 - \lambda a_{31}\Gamma_0 - a_{23}\Theta_3 + a_{33}\Gamma_0)]/D_B,\\
&Y_{23} = -\mathrm{Im}\{[\lambda p_3\Gamma_0 - \lambda\Gamma_2 + a_{23}]/D_B\},\\
&\hat{Y}_{12} = \mathrm{Re}\{[\lambda p_3\Theta_1 + a_{13}\Gamma + \Gamma_1 - \rho v^2(a_{22}\Theta_0 - a_{12}\Gamma_0 + a_{23}\Theta_1 - a_{13}\Theta_2)]/D_B\},\\
&\hat{Y}_{13} = \mathrm{Re}\{[\lambda p_3\Theta_0 - \lambda\Gamma_1 + a_{13} - \rho v^2(\lambda a_{12}\Gamma_0 - \lambda a_{22}\Theta_0 + a_{23}\Theta_0 - a_{13}\Gamma_0)]/D_B\},
\end{aligned} \tag{6.6}
$$

where $D_B = 1 + \lambda(\Gamma + p_3\Theta) + \rho v^2(\Gamma_0 + \lambda\Theta_2 - a_{23}\Theta)$ and

$$
\begin{aligned}
&\Theta_0 = (c_{22}c_{56} - c_{66}^*c_{24})\Lambda_1 + (c_{24}^2 - c_{22}c_{44})\Omega_3 + (c_{24}c_{56} - c_{66}^*c_{44})\Omega_1,\\
&\Theta_1 = \theta_1\theta_2(c_{22}c_{56} - c_{66}^*c_{24})\Omega_1 + (c_{24}^2 - c_{22}c_{44})\Pi_3 + (c_{24}c_{56} - c_{66}^*c_{44})\Pi_1,\\
&\Theta_2 = \theta_1\theta_2(c_{66}c_{24}\Omega_2 - c_{56}c_{12}\Omega_0) + (c_{66}c_{44} + c_{12}c_{44} - c_{24}c_{14})\Pi_2 - c_{56}c_{14}\Pi_0,\\
&\Theta_3 = -c_{66}c_{22}\Lambda_2 + c_{66}^*c_{12}\Lambda_0 + (c_{22}c_{14} - c_{12}c_{24} - c_{66}c_{24})\Omega_2 + c_{66}^*c_{14}\Omega_0,\\
&\Gamma_0 = c_{66}c_{24}\Lambda_2 - c_{56}c_{12}\Lambda_0 + (c_{66}c_{44} + c_{12}c_{44} - c_{24}c_{14})\Omega_2 - c_{56}c_{14}\Omega_0,\\
&\Gamma_1 = (c_{22}c_{56} - c_{66}^*c_{24})p_1p_2\Lambda_0 + (c_{24}^2 - c_{22}c_{44})p_1p_2\Omega_2 + (c_{24}c_{56} - c_{66}^*c_{44})p_1p_2\Omega_0,\\
&\Gamma_2 = c_{66}c_{24}p_1p_2\Lambda_1 - c_{56}c_{12}\Lambda_4 + (c_{66}c_{44} + c_{12}c_{44} - c_{24}c_{14})p_1p_2\Omega_1 - c_{56}c_{14}\Omega_4,\\
&\Gamma_3 = -c_{66}c_{22}p_1p_2\Lambda_1 + c_{66}^*c_{12}\Lambda_4 + (c_{22}c_{14} - c_{12}c_{24} - c_{66}c_{24})p_1p_2\Omega_1 + c_{66}^*c_{14}\Omega_4,
\end{aligned} \tag{6.7}
$$

and parameters Θ, Γ, Ω_i $(i = 0,1,2,3,4)$, Λ_i $(i = 0,1,2,4)$, and Π_i $(i = 0,1,2,3)$ are defined by Eq. (4.5). Here we emphasize again that the explicit formulae (Eq. (6.5)) obtained for the elements of matrices $\boldsymbol{L}(v)$, $\boldsymbol{S}(v)$ and

$\boldsymbol{H}(v)$ take the same form as those for elastosatics (Liou and Sung, submitted for publication) and the parameters expressed in Eqs. (6.6) and (6.7) will be reduced to those for elastostatics when $v = 0$.

7. Monoclinic materials with the symmetry plane at $x_2 = 0$

For monoclinic materials with the symmetry plane at $x_2 = 0$, the elastic stiffness can be obtained from general anisotropic materials by letting $c_{14} = c_{16} = c_{24} = c_{26} = c_{45} = c_{56} = 0$. With these vanishing elements, the sextic equation in Eq. (5.6) becomes

$$\delta = \delta_6 p^6 + \delta_4 p^4 + \delta_2 p^2 + \delta_0 = 0, \tag{7.1}$$

where

$$\begin{aligned}
\delta_6 &= c_{66}c_{22}c_{44} - c_{22}c_{46}^2,\\
\delta_4 &= 2c_{12}c_{25}c_{46} - 2c_{15}c_{22}c_{46} - 2c_{12}c_{44}c_{66} + c_{11}^*c_{22}c_{44} + c_{55}^*c_{22}c_{66}\\
&\quad + 2c_{12}c_{46}^2 - c_{66}c_{25}^2 - c_{44}c_{12}^2 + c_{44}c_{66}(c_{66}^* - c_{66}) - c_{46}^2(c_{66}^* - c_{66}),\\
\delta_2 &= 2c_{15}c_{66}c_{25} + 2c_{12}c_{25}c_{15} + 2c_{15}c_{12}c_{46} - 2c_{55}^*c_{12}c_{66} - 2c_{11}^*c_{25}c_{46} - c_{22}c_{15}^2 - c_{55}^*c_{12}^2\\
&\quad + c_{11}^*c_{22}c_{55}^* + c_{11}^*c_{44}c_{66}^* - c_{11}^*c_{46}^2 - c_{11}^*c_{25}^2 + c_{66}c_{55}^*(c_{66}^* - c_{66}) - 2c_{15}c_{46}(c_{66}^* - c_{66}),\\
\delta_0 &= c_{11}^*c_{55}^*c_{66}^* - c_{66}^*c_{15}^2.
\end{aligned} \tag{7.2}$$

The types of the roots corresponding to Eq. (7.1) are exactly the same as those in Eqs. (6.3a,b). Matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ for this material may be specialized from the general results constructed in Section 3 with the parameters in Eq. (3.6) simplified to

$$\begin{aligned}
\kappa_{11}(p_k) &= c_{22}c_{46}p_k^2 - c_{66}^*c_{25},\\
\kappa_{10}(p_k) &= -c_{22}c_{44}p_k^3 + (c_{25}c_{46} - c_{66}^*c_{44} + c_{46}^2)p_k,\\
\kappa_{21}(p_k) &= (c_{66}c_{25} - c_{12}c_{46})p_k,\\
\kappa_{20}(p_k) &= (c_{66}c_{44} + c_{12}c_{44} - c_{25}c_{46} - c_{46}^2)p_k^2, \quad (k = 1,2,3),\\
\kappa_{31}(p_k) &= -c_{66}c_{22}p_k^2 + c_{66}^*c_{12},\\
\kappa_{30}(p_k) &= c_{22}c_{46}p_k^3 + [c_{46}(c_{66}^* - c_{66}) - c_{12}c_{46}]p_k,\\
\Delta(p_k) &= (c_{66}c_{44} - c_{46}^2)c_{22}p_k^3 + [(c_{46}^2 - c_{44}c_{66}^*)c_{12} + c_{46}c_{25}(c_{66}^* - c_{66})]p_k.
\end{aligned} \tag{7.3}$$

Similarly, the parameters $\theta_k(p_k)$, $(k = 1,2)$ and $\lambda(p_3)$ given by Eqs. (3.13) and (3.15), respectively, may be simplified but with new expressions for $\eta(p_3)$, $\zeta(p_3)$, $\xi(p_k)$, $\beta(p_k)$, $(k = 1,2)$ which are given in Appendix A. Employing the same procedure used as before and the properties of the eigenvalues in Eq. (6.3), matrices $\boldsymbol{L}^{-1}(v)$, $\boldsymbol{S}(v)\boldsymbol{L}^{-1}(v)$, $\boldsymbol{L}(v)$, $\boldsymbol{S}(v)$ and $\boldsymbol{H}(v)$ are obtained as

$$\begin{aligned}
&\boldsymbol{L}^{-1}(v) = \begin{bmatrix} Y_{11} & 0 & Y_{13} \\ 0 & Y_{22} & 0 \\ Y_{13} & 0 & Y_{33} \end{bmatrix}, \quad \boldsymbol{S}(v)\boldsymbol{L}^{-1}(v) = \begin{bmatrix} 0 & -\hat{Y}_{12} & 0 \\ \hat{Y}_{12} & 0 & -\hat{Y}_{23} \\ 0 & \hat{Y}_{23} & 0 \end{bmatrix},\\
&\boldsymbol{L}(v) = \begin{bmatrix} \frac{Y_{33}}{Y_{11}Y_{33}-Y_{13}^2} & 0 & \frac{-Y_{13}}{Y_{11}Y_{33}-Y_{13}^2} \\ 0 & \frac{1}{Y_{22}} & 0 \\ \frac{-Y_{13}}{Y_{11}Y_{33}-Y_{13}^2} & 0 & \frac{Y_{11}}{Y_{11}Y_{33}-Y_{13}^2} \end{bmatrix}, \quad \boldsymbol{S}(v) = \begin{bmatrix} 0 & \frac{-\hat{Y}_{12}}{Y_{22}} & 0 \\ \frac{\hat{Y}_{12}Y_{33}+\hat{Y}_{23}Y_{13}}{Y_{11}Y_{33}-Y_{13}^2} & 0 & -\frac{\hat{Y}_{12}Y_{13}+\hat{Y}_{23}Y_{11}}{Y_{11}Y_{33}-Y_{13}^2} \\ 0 & \frac{\hat{Y}_{23}}{Y_{22}} & 0 \end{bmatrix},\\
&\boldsymbol{H}(v) = \begin{bmatrix} Y_{11} - \frac{\hat{Y}_{12}^2}{Y_{22}} & 0 & Y_{13} + \frac{\hat{Y}_{12}\hat{Y}_{23}}{Y_{22}} \\ 0 & Y_{22} - \frac{\hat{Y}_{12}^2Y_{33}+2\hat{Y}_{12}\hat{Y}_{23}Y_{13}+\hat{Y}_{23}^2Y_{11}}{Y_{11}Y_{33}-Y_{13}^2} & 0 \\ Y_{13} + \frac{\hat{Y}_{12}\hat{Y}_{23}}{Y_{22}} & 0 & Y_{33} - \frac{\hat{Y}_{23}^2}{Y_{22}} \end{bmatrix},
\end{aligned} \tag{7.4}$$

where

$$\begin{aligned}
&Y_{11} = \mathrm{i}[\Theta_0 + \lambda\Theta_1 - a_{13}\Theta]/D_B, \quad Y_{22} = \mathrm{i}[\lambda p_3\Theta_2 + a_{23}\Gamma + \Gamma_2]/D_B,\\
&Y_{33} = \mathrm{i}[\lambda p_3\Theta_3 - \lambda\Gamma_3 + a_{33} + \rho v^2(\lambda a_{21}\Theta_3 - \lambda a_{31}\Gamma_0 - a_{23}\Theta_3 + a_{33}\Gamma_0)]/D_B,\\
&Y_{13} = -\mathrm{Im}\{[\lambda p_3\Theta_0 - \lambda\Gamma_1 + a_{13} - \rho v^2(\lambda a_{12}\Gamma_0 - \lambda a_{22}\Theta_0 + a_{23}\Theta_0 - a_{13}\Gamma_0)]/D_B\},\\
&\hat{Y}_{12} = \mathrm{Re}\{[\lambda p_3\Theta_1 + a_{13}\Gamma + \Gamma_1 - \rho v^2(a_{22}\Theta_0 - a_{12}\Gamma_0 + a_{23}\Theta_1 - a_{13}\Theta_2)]/D_B\},\\
&\hat{Y}_{23} = \mathrm{Re}\{[\lambda p_3\Gamma_0 - \lambda\Gamma_2 + a_{23}]/D_B\},
\end{aligned} \tag{7.5}$$

where $D_B = 1 + \lambda(\Gamma + p_3\Theta) + \rho v^2(\Gamma_0 + \lambda\Theta_2 - a_{23}\Theta)$ and

$$\begin{aligned}
&\Theta_0 = c_{22}c_{46}\Lambda_2 - c^*_{66}c_{25}\Lambda_0 - c_{22}c_{44}\Omega_3 + (c_{25}c_{46} - c^*_{66}c_{44} + c^2_{46})\Omega_1,\\
&\Theta_1 = \theta_1\theta_2(c_{22}c_{46}\Omega_2 - c^*_{66}c_{25}\Omega_0) - c_{22}c_{44}\Pi_3 + (c_{25}c_{46} - c^*_{66}c_{44} + c^2_{46})\Pi_1,\\
&\Theta_2 = \theta_1\theta_2(c_{66}c_{25} - c_{12}c_{46})\Omega_1 + (c_{66}c_{44} + c_{12}c_{44} - c_{25}c_{46} - c^2_{46})\Pi_2,\\
&\Theta_3 = -c_{66}c_{22}\Lambda_2 + c^*_{66}c_{12}\Lambda_0 + c_{22}c_{46}\Omega_3 + [c_{46}(c^*_{66} - c_{66}) - c_{12}c_{46}]\Omega_1,\\
&\Gamma_0 = (c_{66}c_{25} - c_{12}c_{46})\Lambda_1 + (c_{66}c_{44} + c_{12}c_{44} - c_{25}c_{46} - c^2_{46})\Omega_2,\\
&\Gamma_1 = c_{22}c_{46}p_1p_2\Lambda_1 - c^*_{66}c_{25}\Lambda_4 - c_{22}c_{44}p_1p_2\Omega_2 + (c_{25}c_{46} - c^*_{66}c_{44} + c^2_{46})p_1p_2\Omega_0,\\
&\Gamma_2 = (c_{66}c_{25} - c_{12}c_{46})p_1p_2\Lambda_0 + (c_{66}c_{44} + c_{12}c_{44} - c_{25}c_{46} - c^2_{46})p_1p_2\Omega_1,\\
&\Gamma_3 = -c_{66}c_{22}p_1p_2\Lambda_1 + c^*_{66}c_{12}\Lambda_4 + c_{22}c_{46}p_1p_2\Omega_2 + [c_{46}(c^*_{66} - c_{66}) - c_{12}c_{46}]p_1p_2\Omega_0,
\end{aligned} \tag{7.6}$$

and parameters Θ, Γ, Ω_i $(i = 0,1,2,3)$, Λ_i $(i = 0,1,2,4)$, and Π_i $(i = 1,2,3)$ are defined by Eq. (4.5). Note again that by setting $v = 0$ in the expressions expressed in Eq. (7.5), the results for elasostatics are recovered for monoclinic materials with the symmetry plane at $x_2 = 0$ (Liou and Sung, submitted for publication).

8. The explicit expressions of A, B and $L(v)$, $S(v)$, $H(v)$ for orthotropic material

For the orthotropic materials, explicit forms of the elements of $\boldsymbol{L}(v)$, $\boldsymbol{S}(v)$ and $\boldsymbol{H}(v)$ expressed in terms of elastic stiffness have been presented by Dongye and Ting (1989). To validate our results, we show below that our results for general anisotropic materials may be reduced to those of Dongye and Ting (1989) for orthotropic materials. For orthotropic materials the elastic stiffness can be obtained from the monoclinic materials with the symmetry plane at $x_3 = 0$ by letting $c_{16} = c_{26} = c_{45} = 0$, and the associated sextic equation is separated into two equations, i.e.,

$$(c_{44}p^2 + c^*_{55}) = 0, \tag{8.1a}$$

$$c_{22}c_{66}p^4 + [c^*_{11}c_{22} + c_{66}c^*_{66} - (c_{12} + c_{66})^2]p^2 + c^*_{11}c^*_{66} = 0. \tag{8.1b}$$

The roots corresponding to Eq. (8.1) can be classified into two types (Dongye and Ting, 1989). The type I are

$$p_1 = \mathrm{i}n_1, \quad p_2 = \mathrm{i}n_2, \quad p_3 = \mathrm{i}\left(\frac{c^*_{55}}{c_{44}}\right)^{1/2}, \tag{8.2a}$$

and the type II are

$$p_1 = m_1 + \mathrm{i}n_1, \quad p_2 = -m_1 + \mathrm{i}n_1, \quad p_3 = \mathrm{i}\left(\frac{c^*_{55}}{c_{44}}\right)^{1/2}. \tag{8.2b}$$

Elements matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ for orthotropic materials may be simplified directly from the results of monoclinic materials with the symmetry plane at $x_3 = 0$. The forms of the eigenvectors $\boldsymbol{a}_k$ and $\boldsymbol{b}_k$ for orthotropic materials are kept the same as those in Eqs. (5.2) and (5.3), however, the elements a_{ik} $(i,k = 1,2)$ and a_{33} in Eq. (5.4) should be replaced by

$$a_{1k}(p_k) = -(c_{22}p_k^2 + c^*_{66})/(c_{66}c_{22}p_k^2 - c^*_{66}c_{12}), \quad (k = 1,2), \tag{8.3a}$$

$$a_{2k}(p_k) = (c_{12} + c_{66})p_k/(c_{66}c_{22}p_k^2 - c^*_{66}c_{12}), \quad (k = 1,2), \tag{8.3b}$$

$$a_{33} = -\mathrm{i}(c_{44}c_{55}^*)^{-1/2}. \tag{8.3c}$$

The matrices $\boldsymbol{L}^{-1}(v)$, $\boldsymbol{S}(v)\boldsymbol{L}^{-1}(v)$, $\boldsymbol{L}(v)$, $\boldsymbol{S}(v)$ and $\boldsymbol{H}(v)$ in Eq. (5.9) are then calculated, with the properties of the eigenvalues in Eq. (8.2). The results are

$$\begin{aligned}
&\boldsymbol{L}^{-1}(v) = \begin{bmatrix} Y_{11} & 0 & 0 \\ 0 & Y_{22} & 0 \\ 0 & 0 & Y_{33} \end{bmatrix}, \quad \boldsymbol{S}(v)\boldsymbol{L}^{-1}(v) = \begin{bmatrix} 0 & -\hat{Y}_{12} & 0 \\ \hat{Y}_{12} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \\
&\boldsymbol{L}(v) = \begin{bmatrix} Y_{11}^{-1} & 0 & 0 \\ 0 & Y_{22}^{-1} & 0 \\ 0 & 0 & Y_{33}^{-1} \end{bmatrix}, \quad \boldsymbol{S}(v) = \begin{bmatrix} 0 & -\hat{Y}_{12}Y_{22}^{-1} & 0 \\ \hat{Y}_{12}Y_{11}^{-1} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \\
&\boldsymbol{H}(v) = \begin{bmatrix} (Y_{11}Y_{22} - \hat{Y}_{12}^2)Y_{22}^{-1} & 0 & 0 \\ 0 & (Y_{11}Y_{22} - \hat{Y}_{12}^2)Y_{11}^{-1} & 0 \\ 0 & 0 & Y_{33} \end{bmatrix},
\end{aligned} \tag{8.4}$$

where

$$\begin{aligned}
Y_{22} &= -\frac{c_{66}}{c_{66}^*}(p_1p_2)Y_{11} = \mathrm{i}c_{22}c_{66}(c_{12}+c_{66})(p_1p_2)(p_1+p_2)X^{-1}, \\
\hat{Y}_{12} &= [(c_{12}+\rho v^2)c_{66}c_{66}^* - c_{22}^2c_{66}(p_1p_2)^2 - c_{66}^*c_{22}c_{66}(p_1^2+p_2^2) \\
&\quad - c_{22}c_{66}(c_{12}+c_{66})(p_1p_2)]X^{-1}, \\
Y_{33} &= (c_{44}c_{55}^*)^{-1/2},
\end{aligned} \tag{8.5}$$

where

$$X = c_{22}c_{66}[c_{22}c_{66}(p_1p_2)^2 - c_{12}c_{66}^*(p_1^2+p_2^2)] + c_{12}^2c_{66}^*c_{66}^* + \rho v^2(c_{12}+c_{66})(p_1p_2c_{22}c_{66} + c_{12}c_{66}^*). \tag{8.6}$$

The above explicit expressions for the matrices $\boldsymbol{L}(v)$, $\boldsymbol{S}(v)$ and $\boldsymbol{H}(v)$ are slightly different from those given by Dongye and Ting (1989). However, it is noted that for orthotropic materials the two roots p_1 and p_2 in Eq. (8.2), have the relationship (Ting, 1996)

$$\begin{aligned}
&p_1p_2 = -\left(\frac{c_{11}^*c_{66}^*}{c_{22}c_{66}}\right)^{1/2}, \quad p_1^2+p_2^2 = -\frac{(c_{11}^*c_{22}+c_{66}^*c_{66}) - (c_{12}+c_{66})^2}{c_{22}c_{66}}, \\
&p_1+p_2 = \mathrm{i}\left[\frac{(\sqrt{c_{11}^*c_{22}} + \sqrt{c_{66}^*c_{66}})^2 - (c_{12}+c_{66})^2}{c_{22}c_{66}}\right]^{1/2}.
\end{aligned} \tag{8.7}$$

With these results, the elements Y_{11}, Y_{22} and $\hat{Y}_{12}$ in matrices $\boldsymbol{L}(v)$, $\boldsymbol{S}(v)$ and $\boldsymbol{H}(v)$ (Eq. (8.4)) can be further rewritten completely in terms of elastic stiffness as follows:

$$\begin{aligned}
Y_{22} &= \left(\frac{c_{11}^*c_{66}}{c_{66}^*c_{22}}\right)^{1/2} Y_{11} = (c_{66}+c_{12})\sqrt{c_{11}^*c_{66}^*}\left[(\sqrt{c_{11}^*c_{22}} + \sqrt{c_{66}^*c_{66}})^2 - (c_{12}+c_{66})^2\right]^{1/2}X^{-1}, \\
\hat{Y}_{12} &= [(c_{66}^*c_{11}^*c_{22}c_{66})^{1/2} - c_{12}c_{66}^*](c_{12}+c_{66})X^{-1},
\end{aligned} \tag{8.8}$$

where

$$X = (c_{66}+c_{12})[c_{66}^*(c_{11}^*c_{22} - c_{12}^2) - \rho v^2(c_{66}^*c_{11}^*c_{66}c_{22})^{1/2}], \tag{8.9}$$

which completely agree with those elements of $\boldsymbol{L}(v)$, $\boldsymbol{S}(v)$ and $\boldsymbol{H}(v)$ presented by Dongye and Ting (1989).

9. Conclusions

In this paper, the explicit expressions of the three matrices $\boldsymbol{L}(v)$, $\boldsymbol{S}(v)$ and $\boldsymbol{H}(v)$ are presented in terms of the elastic stiffness. Results for the monoclinic materials with symmetry plane at $x_3 = 0$, $x_2 = 0$, and $x_1 = 0$ are all deduced. Results for the case of orthotropic materials appearing in the literature may be recovered from the present results. Final remark is that the vanishing of the determinant of the matrix $\boldsymbol{B}$ constructed in Section 3 will give rise to the secular equation for the investigation of the speed of the surface wave propagating in anisotropic materials. In view of the parameters entering into the matrix $\boldsymbol{B}$ all have been expressed in terms of the elastic stiffness in Section 3, thus the secular equation obtained from $|\boldsymbol{B}| = 0$ will also be expressed in terms of the elastic stiffness for general anisotropic materials. The results of this analysis will be presented elsewhere.

Appendix A

In Eqs. (3.13) and (3.15) θ_k and λ for general anisotropic materials are determined as

$$\theta_k(p_k) = -\frac{\beta(p_k)}{\xi(p_k)}, \quad (k = 1, 2) \tag{A.1}$$

and

$$\lambda(p_3) = \frac{\eta(p_3)}{\zeta(p_3)}, \tag{A.2}$$

respectively, where

$$\beta(p_k) = \sum_{n=0}^{4} \beta_n p_k^n, \quad \xi(p_k) = \sum_{n=0}^{4} \xi_n p_k^n, \quad \eta(p_k) = \sum_{n=0}^{3} \eta_n p_k^n, \quad \zeta(p_k) = \sum_{n=0}^{5} \zeta_n p_k^n \tag{A.3}$$

and

$$\begin{aligned}
\beta_4 &= c_{56}c_{22}c_{44} - c_{46}c_{22}c_{45} + c_{46}c_{25}c_{24} - c_{25}c_{26}c_{44} + c_{24}c_{26}c_{45} - c_{56}c_{24}^2, \\
\beta_3 &= c_{14}c_{25}c_{24} - c_{14}c_{22}c_{45} - c_{25}c_{12}c_{44} - c_{46}c_{26}c_{45} + c_{25}c_{46}^2 - c_{55}^*c_{22}c_{46} \\
&\quad + c_{24}c_{12}c_{45} + c_{24}c_{66}c_{45} + c_{56}c_{22}c_{45} + c_{15}c_{22}c_{44} - c_{15}c_{24}^2 + c_{55}^*c_{26}c_{24} \\
&\quad - c_{56}c_{25}c_{24} - c_{25}c_{26}c_{45} - c_{56}c_{24}c_{46} + c_{56}c_{26}c_{44} - c_{25}c_{66}c_{44} + c_{46}c_{25}^2, \\
\beta_2 &= -c_{25}c_{12}c_{45} - c_{25}c_{16}c_{44} + c_{24}c_{16}c_{45} + c_{14}c_{25}c_{46} + c_{56}c_{25}c_{46} \\
&\quad + c_{14}c_{24}c_{56} - c_{56}c_{12}c_{44} - 2c_{14}c_{26}c_{45} - 2c_{15}c_{24}c_{46} + 2c_{15}c_{26}c_{44} \\
&\quad + c_{56}c_{26}c_{45} - c_{15}c_{25}c_{24} - c_{25}c_{66}c_{45} + c_{46}c_{12}c_{45} + c_{15}c_{22}c_{45} \\
&\quad + c_{55}^*c_{12}c_{24} + c_{55}^*c_{66}c_{24} - c_{55}^*c_{22}c_{14} - c_{55}^*c_{26}c_{46} + c_{14}c_{25}^2 - c_{24}c_{56}^2 \\
&\quad + c_{56}c_{44}(c_{66}^* - c_{66}) - c_{46}c_{45}(c_{66}^* - c_{66}), \\
\beta_1 &= c_{46}c_{16}c_{45} - c_{25}c_{16}c_{45} - c_{15}c_{24}c_{56} + c_{55}^*c_{16}c_{24} - c_{14}c_{66}^*c_{45} \\
&\quad + 2c_{15}c_{26}c_{45} + 2c_{56}c_{25}c_{14} - 2c_{55}^*c_{26}c_{14} + c_{55}^*c_{46}c_{12} + c_{15}c_{66}^*c_{44} \\
&\quad + c_{14}c_{46}c_{56} - c_{56}c_{12}c_{45} - c_{15}c_{25}c_{46} - c_{56}c_{16}c_{44} - c_{15}c_{46}^2 \\
&\quad - c_{55}^*c_{46}(c_{66}^* - c_{66}) + c_{56}c_{45}(c_{66}^* - c_{66}), \\
\beta_0 &= c_{15}c_{66}^*c_{45} + c_{55}^*c_{16}c_{46} - c_{55}^*c_{66}^*c_{14} - c_{56}c_{16}c_{45} - c_{15}c_{46}c_{56} + c_{14}c_{56}^2, \\
\xi_4 &= c_{66}c_{22}c_{44} + 2c_{26}c_{24}c_{46} - c_{44}c_{26}^2 - c_{22}c_{46}^2 - c_{66}c_{24}^2, \\
\xi_3 &= 2c_{26}c_{24}c_{56} - 2c_{66}c_{25}c_{24} + 2c_{26}c_{25}c_{46} + 2c_{66}c_{22}c_{45} - 2c_{56}c_{46}c_{22} - 2c_{45}c_{26}^2 \\
&\quad + c_{26}c_{24}c_{14} - c_{14}c_{46}c_{22} + c_{12}c_{24}c_{46} + c_{16}c_{22}c_{44} - c_{12}c_{26}c_{44} - c_{16}c_{24}^2 \\
&\quad + c_{46}c_{24}(c_{66}^* - c_{66}) - c_{44}c_{26}(c_{66}^* - c_{66}), \\
\xi_2 &= 2c_{56}c_{26}c_{25} + 2c_{16}c_{22}c_{45} - 2c_{12}c_{26}c_{45} - 2c_{16}c_{25}c_{24} - c_{12}c_{66}^*c_{44}
\end{aligned}$$

$$
\begin{aligned}
&+ c_{15}c_{26}c_{24} + c_{12}c_{25}c_{46} - c_{14}c_{56}c_{22} - c_{15}c_{22}c_{46} + c_{66}^{*}c_{24}c_{14}\\
&+ c_{16}c_{26}c_{44} + c_{12}c_{24}c_{56} - c_{26}c_{46}c_{14} - c_{16}c_{24}c_{46} + c_{55}^{*}c_{66}c_{22}\\
&+ c_{26}c_{25}c_{14} + c_{12}c_{46}^{2} - c_{66}c_{25}^{2} - c_{22}c_{56}^{2} - c_{55}^{*}c_{26}^{2}\\
&+ c_{56}c_{24}(c_{66}^{*} - c_{66}) + c_{46}c_{25}(c_{66}^{*} - c_{66}) - 2c_{45}c_{26}(c_{66}^{*} - c_{66}),\\
\xi_1 &= 2c_{12}c_{46}c_{56} + 2c_{16}c_{26}c_{45} - 2c_{12}c_{66}^{*}c_{45} + c_{15}c_{26}c_{25} - c_{15}c_{26}c_{46}\\
&+ c_{14}c_{66}^{*}c_{25} - c_{14}c_{26}c_{56} - c_{16}c_{24}c_{56} - c_{55}^{*}c_{26}c_{12} + c_{55}^{*}c_{16}c_{22}\\
&+ c_{15}c_{66}^{*}c_{24} + c_{25}c_{56}c_{12} - c_{15}c_{22}c_{56} - c_{16}c_{25}c_{46} - c_{16}c_{25}^{2}\\
&+ c_{56}c_{25}(c_{66}^{*} - c_{66}) - c_{55}^{*}c_{26}(c_{66}^{*} - c_{66}),\\
\xi_0 &= c_{55}^{*}c_{16}c_{26} - c_{55}^{*}c_{66}^{*}c_{12} + c_{15}c_{66}^{*}c_{25} - c_{15}c_{26}c_{56} - c_{56}c_{16}c_{25} + c_{12}c_{56}^{2},\\
\eta_3 &= c_{16}c_{26}c_{24} - c_{16}c_{22}c_{46} - c_{12}c_{66}c_{24} + c_{12}c_{26}c_{46} + c_{14}c_{66}c_{22} - c_{14}c_{26}^{2}\\
&+ c_{66}c_{24}(c_{66}^{*} - c_{66}) - c_{46}c_{26}(c_{66}^{*} - c_{66}),\\
\eta_2 &= c_{14}c_{16}c_{22} + c_{12}c_{56}c_{26} + c_{16}c_{26}c_{25} - c_{16}c_{22}c_{56} - c_{12}c_{66}c_{25} - c_{15}c_{26}^{2}\\
&+ c_{15}c_{66}c_{22} - c_{14}c_{26}c_{12} - c_{12}c_{16}c_{24} + c_{46}c_{12}^{2} - c_{11}^{*}c_{22}c_{46} + c_{11}^{*}c_{26}c_{24}\\
&+ c_{66}c_{25}(c_{66}^{*} - c_{66}) - c_{46}c_{12}(c_{66}^{*} - c_{66}) - c_{14}c_{26}(c_{66}^{*} - c_{66})\\
&+ 2c_{16}c_{24}(c_{66}^{*} - c_{66}) - c_{56}c_{26}(c_{66}^{*} - c_{66}),\\
\eta_1 &= c_{15}c_{16}c_{22} + c_{16}c_{46}c_{12} - c_{12}c_{16}c_{25} + c_{56}c_{12}^{2} - c_{15}c_{26}c_{12} + c_{14}c_{16}c_{26} - c_{24}c_{16}^{2}\\
&+ c_{11}^{*}c_{66}^{*}c_{24} + c_{11}^{*}c_{26}c_{25} - c_{11}^{*}c_{26}c_{46} - c_{11}^{*}c_{22}c_{56} - c_{66}^{*}c_{14}c_{12}\\
&+ 2c_{16}c_{25}(c_{66}^{*} - c_{66}) - c_{15}c_{26}(c_{66}^{*} - c_{66}) - c_{56}c_{12}(c_{66}^{*} - c_{66}),\\
\eta_0 &= c_{16}c_{56}c_{12} + c_{15}c_{16}c_{26} - c_{25}c_{16}^{2} + c_{11}^{*}c_{66}^{*}c_{25} - c_{15}c_{66}^{*}c_{12} - c_{11}^{*}c_{26}c_{56},\\
\zeta_5 &= c_{66}c_{22}c_{44} - c_{44}c_{26}^{2} - c_{22}c_{46}^{2} - c_{66}c_{24}^{2} + 2c_{26}c_{24}c_{46},\\
\zeta_4 &= 2c_{16}c_{22}c_{44} + 2c_{12}c_{24}c_{46} + 2c_{26}c_{24}c_{14} - 2c_{12}c_{26}c_{44} - 2c_{14}c_{46}c_{22} - c_{45}c_{26}^{2}\\
&+ c_{26}c_{25}c_{46} + c_{26}c_{24}c_{56} + c_{66}c_{22}c_{45} - c_{66}c_{25}c_{24} - c_{56}c_{46}c_{22} - 2c_{16}c_{24}^{2},\\
\zeta_3 &= 2c_{12}c_{24}c_{14} - 2c_{12}c_{26}c_{45} - 2c_{26}c_{46}c_{14} - 2c_{16}c_{24}c_{46} + 2c_{12}c_{46}^{2}\\
&+ c_{12}c_{24}c_{56} + c_{12}c_{25}c_{46} + c_{15}c_{26}c_{24} - c_{15}c_{22}c_{46} - c_{14}c_{56}c_{22}\\
&+ 2c_{66}c_{24}c_{14} + 2c_{16}c_{26}c_{44} + 2c_{16}c_{22}c_{45} - 2c_{16}c_{25}c_{24}\\
&+ c_{26}c_{25}c_{14} - 2c_{12}c_{66}c_{44} - c_{44}c_{12}^{2} - c_{22}c_{14}^{2} + c_{11}^{*}c_{22}c_{44} - c_{11}^{*}c_{24}^{2}\\
&+ c_{44}c_{66}(c_{66}^{*} - c_{66}) - c_{46}^{2}(c_{66}^{*} - c_{66}),\\
\zeta_2 &= c_{12}c_{25}c_{14} - c_{16}c_{24}c_{56} - c_{14}c_{26}c_{56} + c_{15}c_{12}c_{24} + c_{14}c_{66}c_{25} - c_{45}c_{12}^{2}\\
&+ 2c_{16}c_{26}c_{45} - c_{15}c_{22}c_{14} - c_{15}c_{26}c_{46} - c_{16}c_{25}c_{46} - 2c_{16}c_{12}c_{44} - 2c_{26}c_{14}^{2}\\
&+ 2c_{16}c_{24}c_{14} - 2c_{12}c_{66}c_{45} + c_{15}c_{66}c_{24} + 2c_{12}c_{46}c_{56} + 2c_{12}c_{46}c_{14}\\
&+ 2c_{11}^{*}c_{26}c_{44} - c_{11}^{*}c_{25}c_{24} - 2c_{11}^{*}c_{24}c_{46} + c_{11}^{*}c_{22}c_{45} - c_{56}c_{46}(c_{66}^{*} - c_{66})\\
&+ 2c_{16}c_{44}(c_{66}^{*} - c_{66}) + c_{66}c_{45}(c_{66}^{*} - c_{66}) - 2c_{14}c_{46}(c_{66}^{*} - c_{66}),\\
\zeta_1 &= c_{15}c_{12}c_{46} + c_{12}c_{56}c_{14} + c_{15}c_{16}c_{24} - c_{44}c_{16}^{2} + c_{16}c_{25}c_{14} + c_{11}^{*}c_{66}^{*}c_{44} + 2c_{11}^{*}c_{26}c_{45}\\
&+ 2c_{16}c_{46}c_{14} - 2c_{15}c_{26}c_{14} - 2c_{16}c_{12}c_{45} - c_{11}^{*}c_{46}^{2} - c_{66}^{*}c_{14}^{2} - c_{11}^{*}c_{24}c_{56} - c_{11}^{*}c_{25}c_{46}\\
&- c_{15}c_{46}(c_{66}^{*} - c_{66}) - c_{56}c_{14}(c_{66}^{*} - c_{66}) + 2c_{16}c_{45}(c_{66}^{*} - c_{66}),\\
\zeta_0 &= c_{15}c_{16}c_{46} + c_{16}c_{56}c_{14} - c_{45}c_{16}^{2} - c_{15}c_{66}^{*}c_{14} - c_{11}^{*}c_{46}c_{56} + c_{11}^{*}c_{66}^{*}c_{45}.
\end{aligned}
\tag{A.4}
$$

For monoclinic materials with the symmetry plane at $x_1 = 0$, (A.4) are simplified to

$$
\begin{aligned}
\beta &= (c_{56}c_{22}c_{44} - c_{56}c_{24}^2)p_k^4 + (c_{14}c_{56}^2 - c_{55}^*c_{66}^*c_{14}) + [c_{56}c_{44}(c_{66}^* - c_{66}) \\
&\quad + c_{55}^*c_{66}c_{24} + c_{14}c_{56}c_{24} - c_{56}c_{44}c_{12} + c_{55}^*c_{24}c_{12} - c_{55}^*c_{22}c_{14} - c_{24}c_{56}^2]p_k^2, \\
\xi &= (c_{66}c_{22}c_{44} - c_{66}c_{24}^2)p_k^4 + (c_{12}c_{56}^2 - c_{12}c_{66}^*c_{55}^*) + (c_{66}c_{22}c_{55}^* - c_{14}c_{56}c_{22} \\
&\quad + c_{12}c_{24}c_{56} - c_{12}c_{66}^*c_{44} + c_{66}^*c_{24}c_{14} - c_{22}c_{56}^2 + c_{56}c_{24}(c_{66}^* - c_{66}))p_k^2, \\
\eta &= [c_{14}c_{22}c_{66} - c_{12}c_{66}c_{24} + c_{24}c_{66}(c_{66}^* - c_{66})]p_k^3 \\
&\quad + [c_{11}^*c_{66}^*c_{24} - c_{11}^*c_{56}c_{22} - c_{14}c_{66}^*c_{12} + c_{56}c_{12}^2 - c_{12}c_{56}(c_{66}^* - c_{66})]p_k, \\
\zeta &= (c_{66}c_{22}c_{44} - c_{66}c_{24}^2)p_k^5 + [c_{12}c_{24}c_{56} + c_{11}^*c_{22}c_{44} - c_{14}c_{56}c_{22} - c_{11}^*c_{24}^2 \\
&\quad + 2c_{66}c_{24}c_{14} + 2c_{12}c_{24}c_{14} - 2c_{12}c_{66}c_{44} - c_{44}c_{12}^2 - c_{22}c_{14}^2 + c_{44}c_{66}(c_{66}^* - c_{66})]p_k^3 \\
&\quad + [-c_{66}^*c_{14}^2 - c_{11}^*c_{56}c_{24} + c_{12}c_{56}c_{14} + c_{11}^*c_{66}^*c_{44} - c_{14}c_{56}(c_{66}^* - c_{66})]p_k.
\end{aligned}
\tag{A.5}
$$

For monoclinic materials with the symmetry plane at $x_2 = 0$, (A.4) are simplified to

$$
\begin{aligned}
\beta &= (c_{15}c_{22}c_{44} + c_{25}c_{46}^2 - c_{25}c_{44}c_{12} - c_{25}c_{44}c_{66} + c_{46}c_{25}^2 - c_{55}^*c_{22}c_{46})p_k^3 \\
&\quad + [c_{15}c_{44}c_{66}^* + c_{55}^*c_{12}c_{46} - c_{15}c_{25}c_{46} - c_{15}c_{46}^2 - c_{46}c_{55}^*(c_{66}^* - c_{66})]p_k, \\
\xi &= (c_{66}c_{22}c_{44} - c_{22}c_{46}^2)p_k^4 + [c_{12}c_{25}c_{46} + c_{55}^*c_{22}c_{66} - c_{15}c_{22}c_{46} - c_{12}c_{66}^*c_{44} \\
&\quad + c_{12}c_{46}^2 - c_{66}c_{25}^2 + c_{25}c_{46}(c_{66}^* - c_{66})]p_k^2 + (c_{15}c_{66}^*c_{25} - c_{12}c_{66}^*c_{55}^*), \\
\eta &= [c_{46}c_{12}^2 - c_{12}c_{66}c_{25} - c_{11}^*c_{22}c_{46} + c_{15}c_{22}c_{66} + c_{25}c_{66}(c_{66}^* - c_{66}) \\
&\quad - c_{12}c_{46}(c_{66}^* - c_{66})]p_k^2 + (c_{11}^*c_{66}^*c_{25} - c_{15}c_{12}c_{66}^*), \\
\zeta &= (c_{44}c_{22}c_{66} - c_{22}c_{46}^2)p_k^5 + [c_{12}c_{25}c_{46} + c_{11}^*c_{22}c_{44} - 2c_{12}c_{44}c_{66} - c_{15}c_{22}c_{46} \\
&\quad + 2c_{12}c_{46}^2 - c_{44}c_{12}^2 - c_{46}^2(c_{66}^* - c_{66}) + c_{44}c_{66}(c_{66}^* - c_{66})]p_k^3 \\
&\quad + [c_{11}^*c_{44}c_{66}^* + c_{15}c_{12}c_{46} - c_{11}^*c_{25}c_{46} - c_{11}^*c_{46}^2 - c_{15}c_{46}(c_{66}^* - c_{66})]p_k.
\end{aligned}
\tag{A.6}
$$

Appendix B

The obtained explicit formulae for the elements of matrices $\boldsymbol{L}(v)$, $\boldsymbol{S}(v)$ and $\boldsymbol{H}(v)$ for general anisotropic materials are exactly the same as those for elaststatics (Liou and Sung, submitted for publication). The explicit formulae for the elements of the matrix $\boldsymbol{L}(v)$ are

$$
\boldsymbol{L}(v) = \begin{bmatrix} (Y_{22}Y_{33} - Y_{23}^2)/D_L & (Y_{13}Y_{23} - Y_{12}Y_{33})/D_L & (Y_{12}Y_{23} - Y_{13}Y_{22})/D_L \\ (Y_{13}Y_{23} - Y_{12}Y_{33})/D_L & (Y_{11}Y_{33} - Y_{13}^2)/D_L & (Y_{12}Y_{13} - Y_{23}Y_{11})/D_L \\ (Y_{12}Y_{23} - Y_{13}Y_{22})/D_L & (Y_{12}Y_{13} - Y_{23}Y_{11})/D_L & (Y_{11}Y_{22} - Y_{12}^2)/D_L \end{bmatrix}, \tag{B.1}
$$

while the explicit formulae for the elements of matrix $\boldsymbol{S}(v)$ are as

$$
\begin{aligned}
S_{11} &= [\hat{Y}_{12}(Y_{12}Y_{33} - Y_{13}Y_{23}) + \hat{Y}_{13}(Y_{13}Y_{22} - Y_{12}Y_{23})]/D_L, \\
S_{12} &= [\hat{Y}_{12}(Y_{13}^2 - Y_{11}Y_{33}) + \hat{Y}_{13}(Y_{11}Y_{23} - Y_{12}Y_{13})]/D_L, \\
S_{13} &= [\hat{Y}_{12}(Y_{11}Y_{23} - Y_{12}Y_{13}) + \hat{Y}_{13}(Y_{12}^2 - Y_{11}Y_{22})]/D_L, \\
S_{21} &= [\hat{Y}_{12}(Y_{22}Y_{33} - Y_{23}^2) + \hat{Y}_{23}(Y_{13}Y_{22} - Y_{12}Y_{23})]/D_L, \\
S_{22} &= [\hat{Y}_{12}(Y_{13}Y_{23} - Y_{12}Y_{33}) + \hat{Y}_{23}(Y_{11}Y_{23} - Y_{12}Y_{13})]/D_L, \\
S_{23} &= [\hat{Y}_{12}(Y_{12}Y_{23} - Y_{13}Y_{22}) + \hat{Y}_{23}(Y_{12}^2 - Y_{11}Y_{22})]/D_L, \\
S_{31} &= [\hat{Y}_{13}(Y_{22}Y_{33} - Y_{23}^2) + \hat{Y}_{23}(Y_{13}Y_{23} - Y_{12}Y_{33})]/D_L, \\
S_{32} &= [\hat{Y}_{13}(Y_{13}Y_{23} - Y_{12}Y_{33}) + \hat{Y}_{23}(Y_{11}Y_{33} - Y_{13}^2)]/D_L, \\
S_{33} &= [\hat{Y}_{13}(Y_{12}Y_{23} - Y_{13}Y_{22}) + \hat{Y}_{23}(Y_{12}Y_{23} - Y_{11}Y_{23})]/D_L.
\end{aligned}
\tag{B.2}
$$

As to matrix $\boldsymbol{H}(v)$, the explicit formulae are

$$
\begin{aligned}
H_{11} &= Y_{11} - [\hat{Y}_{12}^2(Y_{11}Y_{33} - Y_{13}^2) + 2\hat{Y}_{12}\hat{Y}_{13}(Y_{12}Y_{13} - Y_{11}Y_{23}) + \hat{Y}_{13}^2(Y_{11}Y_{22} - Y_{12}^2)]/D_L, \\
H_{12} &= Y_{12} - [\hat{Y}_{12}^2(Y_{12}Y_{33} - Y_{13}Y_{23}) + \hat{Y}_{12}\hat{Y}_{23}(Y_{12}Y_{13} - Y_{11}Y_{23}) \\
&\quad + \hat{Y}_{12}\hat{Y}_{13}(Y_{13}Y_{22} - Y_{12}Y_{23}) + \hat{Y}_{13}\hat{Y}_{23}(Y_{11}Y_{22} - Y_{12}^2)]/D_L, \\
H_{13} &= Y_{13} - [\hat{Y}_{13}^2(Y_{13}Y_{22} - Y_{12}Y_{23}) + \hat{Y}_{12}\hat{Y}_{23}(Y_{13}^2 - Y_{11}Y_{33}) \\
&\quad + \hat{Y}_{12}\hat{Y}_{13}(Y_{12}Y_{33} - Y_{13}Y_{23}) + \hat{Y}_{13}\hat{Y}_{23}(Y_{11}Y_{23} - Y_{12}Y_{13})]/D_L, \\
H_{22} &= Y_{22} - [\hat{Y}_{12}^2(Y_{22}Y_{33} - Y_{23}^2) + 2\hat{Y}_{12}\hat{Y}_{23}(Y_{13}Y_{22} - Y_{12}Y_{23}) + \hat{Y}_{23}^2(Y_{11}Y_{22} - Y_{12}^2)]/D_L, \\
H_{23} &= Y_{23} - [\hat{Y}_{23}^2(Y_{11}Y_{23} - Y_{12}Y_{13}) + \hat{Y}_{12}\hat{Y}_{23}(Y_{13}Y_{23} - Y_{12}Y_{33}) \\
&\quad + \hat{Y}_{12}\hat{Y}_{13}(Y_{22}Y_{33} - Y_{23}^2) + \hat{Y}_{13}\hat{Y}_{23}(Y_{13}Y_{22} - Y_{12}Y_{23})]/D_L, \\
H_{33} &= Y_{33} - [\hat{Y}_{13}^2(Y_{22}Y_{33} - Y_{23}^2) + 2\hat{Y}_{13}\hat{Y}_{23}(Y_{13}Y_{23} - Y_{12}Y_{33}) + \hat{Y}_{23}^2(Y_{11}Y_{33} - Y_{13}^2)]/D_L.
\end{aligned} \tag{B.3}
$$

In Eqs. (B.1)–(B.3), Y_{ij} $(i,j = 1,2,3)$ and $\hat{Y}_{ij}$ $(i, j = 1, 2, 3,\ i \neq j)$ are given in Eq. (4.3), and

$$
D_L = Y_{11}Y_{22}Y_{33} + 2Y_{12}Y_{13}Y_{23} - Y_{11}Y_{23}^2 - Y_{22}Y_{13}^2 - Y_{33}Y_{12}^2. \tag{B.4}
$$

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